THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

19 KIDMORE ROAD, CAVERSHAM, READING

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XVIII. DECEMBER, 1934. No. 231.

A NOTE ON ISOGONAL CONJUGATES.

BY J. CLEMOW.

THE present article gives an easy way of introducing to schoolboys a geometrical transformation which is important in higher geometry and which is of a type different from Inversion, Projection and Reciprocation, with which they are familiar. There is, of course, nothing new in the transformation, which is well known to geometers, but I have never seen its connection with isogonal conjugates mentioned in print.*

We have a point P in the plane of a triangle ABC. Then if we draw a line BP' through B such that angle $P'BA$ is equal to angle PBC, and CP' through C so that angle $P'CA$ is equal to angle PCB, then the similar line drawn through A will concur with the other two in P'. This follows immediately from Ceva's theorem. P' is called the isogonal conjugate of P.

FIG. 1.

We have at once that :

The incentre and the three excentres of ABC are their own isogonal conjugates.

* Since writing this note I have found three references to the work. Duporcq's *Géométrie Moderne*, § 177, gives an interesting approach from another point of view. There is also something on the subject in Baker's *Principles of Geometry*, vol. ii, and in Coolidge's *Circle and Sphere*.

The circumcentre and the orthocentre of ABC are isogonal conjugates.

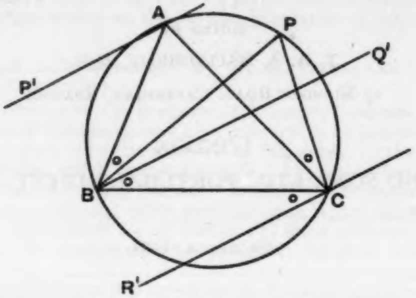


FIG. 2.

The isogonal conjugate of a point P on the circumcircle of ABC is at infinity. For if AP' is the isogonal of AP, BQ' of BP, and CR' of CP, then

$$\angle Q'BC = \angle PBA = \angle PCA = \angle BCR'.$$

Therefore BQ' is parallel to CR'.
Similarly BQ' is parallel to AP'.
Therefore P' the isogonal conjugate of P is at infinity.

Consider a point P on BC. The isogonal conjugate of this is a point on AR the isogonal of AP. Let Q be a point on AP near to P. Then its isogonal is a point Q' on AR near to A. Then as Q approaches P, so Q' approaches A along AR. Thus to *points* of BC correspond *directions* at A, the direction associated with P being that of the isogonal of AP.

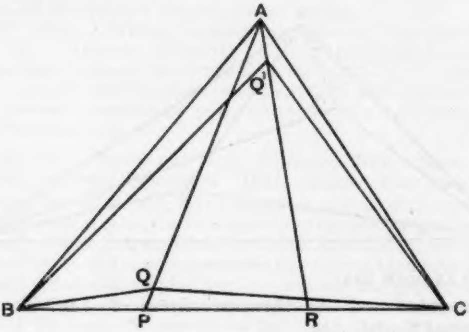


FIG. 3.

We are now in a position to study the results of applying this transformation by isogonal conjugates to various algebraic loci.

A straight line through a vertex becomes another straight line (the isogonal line) through the same vertex.

The circumcircle becomes the line at infinity.

(We assume for the moment that the transformation is algebraic as well as (1, 1). This is almost obvious but will be proved by algebra later.) Algebraic loci transform to algebraic loci.

A general line meets each side of the triangle ABC in one point. Therefore it transforms into a curve through A, B. C. Also a general line through A meets the given general line in one point, not at A. Therefore any line through A meets the transformed locus in one other point besides A. Therefore this locus is of the second degree—a conic—and circumscribes ABC.

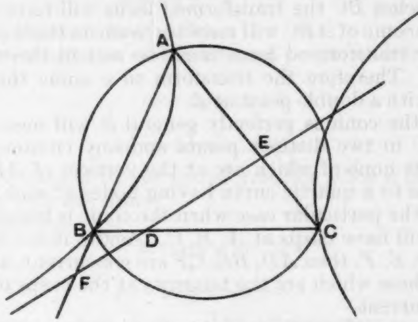


FIG. 4.

To the points where this conic cuts the line at infinity will correspond points where the line meets the circumcircle. Therefore the transformed conic is an ellipse, a parabola or a hyperbola, according as the line does not meet, touches, or intersects the circumcircle of ABC.

Now if two lines through a vertex are inclined at an angle α, their isogonals will be inclined at the same angle α. Consider a line passing through the circumcentre of ABC. Let it meet the circumcircle in Q and R. Then angle QAR is a right angle. Now the transform of the line QR will be a hyperbola, and the isogonals of Q and R will lie on the curve and also on the line at infinity. Therefore since they subtend a right angle at A, which is also on the curve, the hyperbola must be rectangular. So we have the theorem that lines through the circumcentre transform to rectangular hyperbolas circumscribing ABC, and conversely. Also, since the orthocentre is the isogonal of the circumcentre, these rectangular hyperbolas will all pass through the orthocentre of ABC.

The fact that perpendicular lines through A transform to perpendicular lines indicates that the circular points at infinity remain unchanged by the transformation. Or otherwise, the circular points are the intersections of the circumcircle and the line at infinity, but

since these two transform into each other their pair of intersections must transform into the same two points.

A conic through ABC will transform to a straight line.

A conic through B and C will meet AB and AC each in one further point not at A, and a conic through ABC in two points other than B and C, and hence will transform to another conic through B and C.

In particular a circle through B and C, since it is a conic through the circular points at infinity, will transform to a circle through B and C.

A conic through A only will meet BC in two points other than B and C. Hence the transform will pass twice through A, having distinct tangents there if the two points on BC are distinct, but if the conic touches BC the transformed locus will have a cusp at A. Also a circumconic of ABC will meet the conic in three points besides A, hence the transformed locus must be met in three points by a general line. Therefore the transform to a conic through A is a cubic curve with a double point at A.

Finally, if the conic is perfectly general it will meet each of the sides of ABC in two distinct points and any circumconic in four distinct points none of which are at the vertices of ABC. Thus it will transform to a quartic curve having nodes at each of the points A, B, C. In the particular case when the conic is inscribed in ABC, the quartic will have cusps at A, B, C. Again, if the conic touches the sides in D, E, F, then AD, BE, CF are concurrent, and hence the isogonals of these which are the tangents at the cusps of the quartic are also concurrent.

We have now sufficient apparatus to transform theorems about straight lines and conics into theorems about trinodal quartics, and *vice versa*.

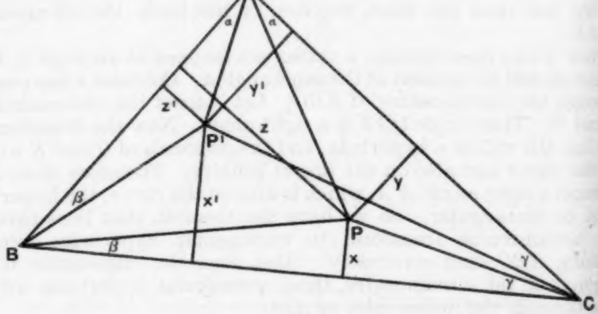


FIG. 5.

For the sake of completeness we now give the algebra.

Let $P(x, y, z)$ and $P'(x', y', z')$ be isogonal conjugates with respect to the triangle of reference ABC, the coordinates being trilinear.

Then

$$\frac{x}{\sin\beta}=\frac{z}{\sin(B-\beta)};\quad \frac{y}{\sin(C-\gamma)}=\frac{x}{\sin\gamma};\quad \frac{y}{\sin\alpha}=\frac{z}{\sin(A-\alpha)}$$

$$\frac{x'}{\sin(B-\beta)}=\frac{z'}{\sin\beta};\quad \frac{y'}{\sin\gamma}=\frac{x'}{\sin(C-\gamma)};\quad \frac{y'}{\sin(A-\alpha)}=\frac{z'}{\sin\alpha}$$

or

$$x:y:z=\frac{1}{x'}:\frac{1}{y'}:\frac{1}{z'},$$

so that the transformation is Cremona's quadratic transformation.

The line at infinity is

$$ax+by+cz=0,$$

and this transforms to

$$a/x+b/y+c/z=0,$$

which is the circumcircle of ABC.

Again, the coordinates of the circular points at infinity are given by the roots of

$$ax+by+cz=0$$

and

$$a/x+b/y+c/z=0,$$

and so are unchanged by the transformation since it merely changes one equation into the other.

Any straight line

$$lx+my+nz=0$$

becomes

$$l/x+m/y+n/z=0,$$

which is a conic. And so on. Any algebraic curve transforms to an algebraic curve.

As an example of the transformation it seems worth while giving the remarkable theorem that

> Circumconics of a triangle orthogonal to the circumcircle at the ends of a diameter through the mid-point of a side touch each other at the vertex corresponding to that side and have as tangent there the symmedian.

Let ABC be the triangle, D the mid-point of BC, and O the centre of the circumcircle of ABC. Then it is easy to prove by elementary geometry that the locus of isogonal conjugates of points of OD is a rectangular hyperbola circumscribed to ABC and with D as centre. That is, D is the intersection of tangents at the two points where the hyperbola meets the line at infinity. Now, when we transform the hyperbola to OD, these points on the line at infinity transform to the points where OD meets the circumcircle of ABC, and the tangents transform to two circumconics touching OD at these two points on the circumcircle. Now these two conics must pass through the transform of D, which is a point on the symmedian through A and in the neighbourhood of A. Hence the theorem.

We may remark that circular inversion is a transformation of the type considered above; the triangle is the centre of the circle and the two circular points at infinity. J. C.

RANDOMISATION, AND AN OLD ENIGMA OF CARD PLAY.

BY R. A. FISHER.

THE process of randomisation has in recent years come to play such a central part in experimental design that it is of some interest to find that it affords a means of resolving one of the oldest paradoxes which arose in discussions of gaming.

Readers of Todhunter's *The Mathematical Theory of Probability* will recall his account (sections 187-190, pp. 106-110) of the correspondence between Montmort and Nicolas Bernoulli on the rule by which the players might guide themselves most advantageously in the game called " le Her ".

In the game, when played by two persons, the dealer A deals himself and his opponent B a single card each, the cards being valued in order from the ace as lowest to the king as highest. First B has the option, if he wishes it, to change his card with that which A holds, but if A holds a king, he is allowed to retain it. Next A, whether the cards are changed or not, has the option of interchanging his with one chosen at random from the pack, but if he draws a king he must retain his original card. It is a convention of the game that if the two cards finally held are equal A is the winner.

It being understood * that B will change any card lower that 7, and will retain any card of higher value, and that, if B does not exercise his option, A will change any card lower than an 8 and retain any higher card, the questions which Montmort's discussion was intended to resolve were whether B should exercise his option when he holds a 7, and whether A should do so when he holds an 8.

The paradoxical point which led to the dispute lies in the facts, which can be shown by simply counting the chances, that if it is B's rule to change a 7, then A will gain by adopting the rule of changing his 8, and *vice versa*. It is to A's advantage to follow a like rule with B. But, if it is A's rule to change his 8, then it is to B's advantage to retain his 7, while, in the contrary case, he would gain by changing it. It is thus to B's advantage to act on a rule unlike that of A. It was Nicolas Bernoulli's view that both players ought to change in the doubtful cases, while Montmort held that no absolute rule could be given.

Montmort's conclusion, though obviously correct for the limited aspect in which he viewed the problem, is unsatisfactory to common sense, which suggests that in all circumstances there must be, according to the degree of our knowledge, at least one rule of conduct which shall be not less satisfactory than any other ; and this his discussion fails to provide. Granted that if B knows A's rule he can most advantageously adopt an unlike rule for himself, this does not answer the question : How should B act if he does not know A's

* Todhunter does not discuss these preliminary points, but speaks of them as " tacitly allowed by the disputants " ; it is, however, demonstrable that these rules are advantageous to the players who follow them.

rule, but feels instead justified in presuming that A's conduct will be guided by his own advantage?

This question, which is left untouched by the discussions of Montmort and Todhunter, may be resolved by the simple consideration that there are more than two rules for B to choose from. He may always change a 7 when he has it, or he may never change it, but, again, he may adopt the rule of changing it occasionally, with a definitely chosen frequency. There will, so far as the mathematical problem is concerned, be nothing to guide him as to when to change and when to retain his card, but without such guidance he may, none the less, adopt such a policy as changing once in every three trials at haphazard, or with any other frequency preferred. The same is true of A, and we may follow out the consequences of the supposition that A chooses a frequency, p, for changing his 8, and B a frequency, p', for changing his 7.

Knowing the manner in which the players exercise their options, it is easy to calculate B's chance of winning in each of the 13×13 ways in which the cards can be originally dealt. Thus, if A has dealt his opponent an ace, and himself a 2, the cards will certainly be changed, and A, receiving an ace, will exercise his option of choosing another from the pack. B's chance of winning rests on the possibility that, out of the 50 cards available, A shall choose either one of the three remaining aces or one of the four kings. His probability of winning is therefore $\frac{7}{50}$. The table below shows the chances out of 50 for all possible cases. There are separate columns, when B receives a 7, for the cases of his changing and retaining it respectively,

TABLE SHOWING NUMBER OF CHANCES OUT OF 50 FAVOURABLE TO B, FOR ALL COMBINATIONS OF CARDS ORIGINALLY HELD.

A's card	B's card 1	2	3	4	5	6	7 p'	7 q'	8	9	10	J	Q	K
1	0	0	0	0	0	0	0	27	31	35	39	43	47	50
2	7	0	0	0	0	0	0	27	31	35	39	43	47	50
3	11	11	0	0	0	0	0	27	31	35	39	43	47	50
4	15	15	15	0	0	0	0	27	31	35	39	43	47	50
5	19	19	19	19	0	0	0	27	31	35	39	43	47	50
6	23	23	23	23	23	0	0	27	31	35	39	43	47	50
7	27	27	27	27	27	27	0	24	31	35	39	43	47	50
8	31	31	31	31	31	31	31	24	28	35	39	43	47	50p
								0	0	50	50	50	50	50q
9	35	35	35	35	35	35	35	0	0	0	50	50	50	50
10	39	39	39	39	39	39	39	0	0	0	0	50	50	50
J	43	43	43	43	43	43	43	0	0	0	0	0	50	50
Q	47	47	47	47	47	47	47	0	0	0	0	0	0	50
K	0	0	0	0	0	0	0	0	0	0	0	0	0	0

and separate lines, when no interchange has been made, for the cases of A changing and retaining his 8. With respect to the frequency of the different cases it should be noted, apart from the frequencies assigned to the exercise of the two options, that the cases in which the opponents have initially cards of different value each occur 16 times in 52×51 trials, while those in which they have the same value occur only 12 times.

In all it appears that, out of 5525 games, B's expectation of winning is $2828 + 6p + 10p' - 16pp'$. If, therefore, he fixes on any frequency p' greater than $\frac{3}{8}$, his opponent, if he, A, acts to his own greatest advantage, will put $p = 1$, so minimising B's expectation at the value $2834 - 6p'$. If, on the other hand, he chooses a value of p' less than $\frac{3}{8}$, his opponent, if he acts to his own advantage, will put $p = 0$, so minimising the expectation at the value $2828 + 10p'$. It is now clear how B should act to his own greatest advantage, for the function (Fig. 1):

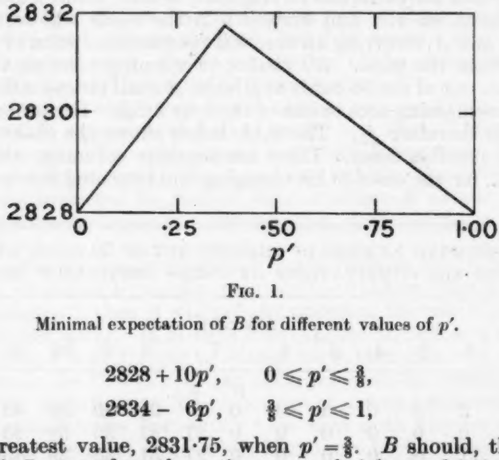


FIG. 1.

Minimal expectation of B for different values of p'.

$$\begin{array}{ll} 2828 + 10p', & 0 \leqslant p' \leqslant \frac{3}{8}, \\ 2834 - \ \ 6p', & \frac{3}{8} \leqslant p' \leqslant 1, \end{array}$$

has its greatest value, 2831·75, when $p' = \frac{3}{8}$. B should, therefore, change his 7 at random three times out of eight, and it is then indifferent to him what policy A pursues.

Equally A may argue that if he chooses a value of p less than $\frac{5}{8}$ he will leave it open to his opponent to put $p' = 1$, so maximising his expectation at $2838 - 10p$. While if he fixes p at a value greater than $\frac{5}{8}$, B may put $p' = 0$, and maximise his expectation at $2828 + 6p$. B's possible advantage is thus minimised at 2831·75 (Fig. 2), when $p = \frac{5}{8}$, and it is then a matter of indifference what policy B adopts. In fact, as a function of p and p', B's expectation is represented by an anticlastic surface, and if each player pursues his own advantage, the chances of the game are stabilised at the saddle.

It is a slightly unexpected feature that, in spite of the game going

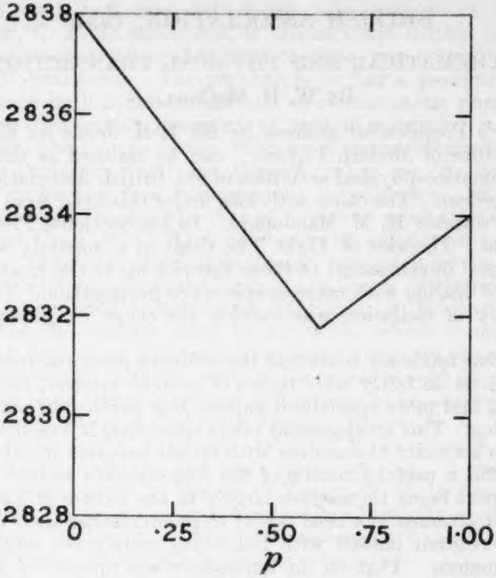


FIG. 2.

Maximal expectation of B for different values of p.

to A when the cards are equal, the chances should be on the whole to the advantage of B. **R. A. F.**

GLEANINGS FAR AND NEAR.

990. A new method of estimating the surface area of living animals used in metabolism work seems likely to be of interest to many biological workers outside the agricultural sphere. Such a method has recently been developed by me in this laboratory.

Although the animals immediately concerned are pigs, the method is applicable generally to all animals not encumbered with a thick shaggy coat, and consists, essentially, in the estimation of the approximate value of the integral $\int P\,dx$ over the surface of the animal, where P is the perimeter in a plane perpendicular to the axis of x. In the case of pigs, it appeared satisfactory to take $P = k\pi(a+b)$, assuming the cross-section elliptical for the head and body. Here k is a constant which indicates the departure from perfect ellipticity and varies somewhat in value at different parts of the body, while a and b have their usual connotations in the approximate formula for the perimeter of an ellipse. The approximate integration is then made by Simpson's rule, the values of the ordinates a and b for different values of x being read off directly from scale photographs, one taken from the side and the other from directly above the animal. The ears, legs, and tail are estimated separately.—*Nature* (Feb. 20, 1932), p. 281. [Per Mr. N. Anning.]

BRITISH ASSOCIATION, 1934.

MATHEMATICAL AND PHYSICAL TRANSACTIONS.

By W. H. McCrea.

This year's Presidential address by Sir J. H. Jeans on the "New World-picture of Modern Physics" can be claimed as the first of the mathematico-physical activities of the British Association meeting in Aberdeen. The other activities under this head were presided over by Professor H. M. Macdonald. In his Sectional Presidential address on "Theories of Light" he dealt in a masterly way with the historical development of these theories up to the most modern methods of dealing with macroscopic wave-propagation. The quantum theory of radiation was outside the scope he prescribed for himself.*

The recent tendency to devote the ordinary sectional transactions to discussions on fairly wide topics of current interest, rather than to isolated and more specialised papers, was particularly marked at this meeting. This arrangement offers specialists in one field opportunities to acquaint themselves with recent progress in other fields, and so fulfils a useful function of the Association's annual meeting.

The papers being themselves largely in the nature of summaries, one cannot attempt in a brief report to re-summarise their contents, but must content oneself with indicating merely the scope of the chief discussions. That on the ionosphere was opened by Professor E. V. Appleton and continued by Messrs. J. A. Ratcliffe and R. Naismith. Ultra-violet light and charged particles from the sun both apparently act as ionising agencies in maintaining the ionosphere, but their relative importance has not yet been determined. The theory of metals received considerable attention. Professor R. H. Fowler opened a discussion on the new quantum electronic theory of metals, which was initiated by Sommerfeld about 1927, and has been so brilliantly successful in subsequent applications. Professor C. G. Darwin expressed the view that a study of this theory provides an excellent means of learning the methods of modern quantum mechanics. Dr. H. Jones reported upon recent developments due to himself and Professor N. F. Mott which provide a theoretical basis for well-known rules relating electrical properties of metals to their valencies and positions in the periodic table. Professor W. L. Bragg in opening a further discussion gave an account of recent work on the structure of alloys, stressing particularly the importance of free electrons for the theory of alloys. Dr. A. J. Bradley described recent experimental work in this field. Professor G. I. Taylor spoke on his theory of plasticity in metals, and showed photographs of an ingenious working model of a crystal lattice in which the atoms are represented by small magnets made to float vertically in water,

* These addresses are reprinted in full in *The Advancement of Science*, 1934, and in the *Annual Report*, 1934. The latter will also contain the paper by Campbell and Paterson mentioned below. *Cf.* also *Nature*, 8th and 29th September, 1934, for the presidential addresses.

some with north and some with south poles uppermost. Messrs. J. C. Slater, G. P. Thomson and H. Jones made further contributions. A discussion on unified field theories was opened by Professor E. T. Whittaker. The problem is to find a geometry which will represent both gravitational and electromagnetic phenomena, just as the Riemannian geometry of general relativity represents gravitational phenomena alone. The most recent attempts, as described by Professor Whittaker, depend on the use of a projective geometry employing five homogeneous coordinates. Dr. W. H. McCrea discussed the introduction of quantum mechanics into this scheme, and Dr. J. H. C. Whitehead gave an account of projective relativity. The same physical problems, treated from an entirely different standpoint, are at the basis of the theory of electric charge and mass expounded subsequently by Sir A. S. Eddington. He showed that the various steps in his work, which is now fairly familiar, all depend on questions in existing quantum theory which require elucidation. Hence, even if his work should prove to be mistaken in parts, these questions still remain to be answered.

In the department of Cosmical Physics, Messrs. E. A. Milne, T. Dunham, O. Struve, J. A. Carroll and E. G. Williams gave accounts of work on stellar spectra and their interpretation. A symposium on telescopes was given by Messrs. C. Young, W. M. H. Greaves and C. R. Burch. A joint discussion with the Chemistry section dealt with "Heavy Hydrogen". The joint sessions with the Engineering section on Technical Physics were a new feature.

Isolated papers were read by R. H. Fowler and G. B. B. M. Sutherland on specific heats of gases, J. M. Stagg on the British Polar Year expedition, J. B. Tait on meteorology, F. W. Aston on the "roll-call of the isotopes", W. H. McCrea on relativity, H. G. Howell on spectroscopy, J. A. Carroll on Fourier transforms, W. L. Marr on Desargues' configurations from a quintic curve, and E. A. Maxwell on the theory of surfaces. Thus the proportion of pure mathematics was unusually small. A paper responding to the demand for a consideration of the relation of science to the life of the community was supplied by Messrs. N. R. Campbell and C. C. Paterson on "Photoelectricity, art and policy: an historical study".

A visit was paid to the Natural Philosophy department of Aberdeen University under the direction of Professor Carroll. The exhibits included a demonstration of Zernike's phase contrast test applied to mirrors and microscopes, arranged by Messrs. C. R. Burch and L. H. J. Phillips. On 10th September a party went to Braemar to witness the unveiling by Princess Arthur of Connaught of the memorial to Johann von Lamont, who was born near there in 1805 and later became Astronomer-Royal to the King of Bavaria.

W. H. McCrea.

---

991. Oh ! je donnerais bien cent sous au mathématicien qui me démontrerait par une equation algébrique l'existence de l'enfer.

Il jeta une pièce en l'air en criant :—Face pour Dieu !—H. de Balzac, *La Peau de Chagrin*. [Per Mr. J. B. Bretherton.]

THE STATISTICAL THEORY OF TURBULENT MOTION.

By G. F. P. Trubridge.

The study of the turbulent motion of fluids is of great importance in aerodynamics and engineering, besides being a fundamental problem of hydrodynamics. Several attempts have been made to develop an adequate theory of turbulent motion, but, except for a few good empirical formulae, little advance has been made. L. Prandtl has given a clear and concise account of his own theory and that of Th. von Kármán in a recent paper (1), which contains full references to the literature of the subject. H. Bateman has also contributed an excellent account of the present state of the theory in the Report on Hydrodynamics of the National Research Council of America. In neither of these accounts, however, has reference been made to the theory put forward by J. M. Burgers of Delft, in a series of papers (2) published in 1929, and further developed in 1933. In these papers Burgers has attempted to apply the methods of classical statistical mechanics to the theory of turbulent fluid motion. In this article a simplified account of his methods will be given.

Burgers considers the completely turbulent state of flow of an incompressible viscous fluid between two parallel walls at a distance h apart. The x-axis is taken in the direction of the main flow, the y-axis perpendicular to the walls, and the z-axis perpendicular to both of the other axes. The investigations are limited to a finite length of the channel in the x-direction, the magnitude of the length being large compared with the breadth of the channel. The motion is considered in two dimensions, so that no account is taken of variations of velocity in the z-direction. The restriction to the two-dimensional case may introduce special features which will not be found in turbulent motion, but there does not yet appear to be any method of attacking the problem in three dimensions.

The usual non-dimensional variables are introduced into the Navier-Stokes hydrodynamical equations, which are assumed to be valid for all types of motion. Following the methods introduced by Reynolds and Lorentz, the real motion is decomposed into the so-called mean flow and the continually fluctuating relative motion. By this method it is hoped to obtain an idea of the structure of the turbulent motion, the connection between the motion in the central part of the channel and the boundary layers in the immediate vicinity of the walls, and the mechanism of the balance of energy between the mean and the relative motions. From the equations developed it is clear that the mean motion is influenced by the relative motions, as in the equations governing the former occur the mean values of the squares and products of the velocity components of the relative motion. The main object of the theory of turbulence is to calculate these mean values, and also a value for the resistance experienced in sending the fluid through the channel.

In applying the methods of statistical mechanics to the problem

it is essential to choose particular "objects" to describe the various states of motion. Burgers assumes that the relative motion can be fully specified by means of a stream function ψ. He introduces a quadratic point lattice, in which the spacing ϵ of the lattice points is assumed to be very small compared with the channel breadth. At any given time the state of flow can be specified by the values of the stream function at the finite number of lattice points. The velocity components are then represented by certain difference equations.

An N-dimensional phase space (the ξ-space) is then introduced, with the values of the stream function at the lattice points as coordinates. Each instantaneous state of flow is then given by a point in this phase space. Now, in classical statistical theory, great importance is attached to the fact that there is no crowding of points together into favoured regions of the phase space. Burgers has not succeeded, however, in proving the validity of this theorem of Liouville's for his phase space, except for the case of an ideal fluid.

Instead of the usual condition of constant total energy, Burgers uses the dissipation condition, which expresses that the relative motion obtains every second just as much energy from the mean motion as it loses in the same time in consequence of the viscous nature of the fluid.

The various states of the relative motion at a series of instants with equal intervals of time between them are represented simultaneously by a number of points M in the phase space. When M is sufficiently large, it is apparent that mean values with respect to the time of quantities depending on the values of the stream function for the relative motion can be calculated by using the points in the ξ-space.

The ξ-space is divided up into a number of small N-dimensional cells and the numbers of points in these cells are denoted by n_1, n_2, n_3, ... The statistical probability W of the ensemble of fields of flow considered is then obtained by analogy with the kinetic theory and is given by the usual formula $W = M!/n_1!\,n_2!\,n_3!\,\ldots$. The hypothesis is then used that an appropriate set of points in the ξ-space, enabling mean values to be calculated, which also satisfies the appropriate dissipation condition and the boundary conditions, makes W a maximum. By using this idea and the equation for the mean motion, an expression for the distribution function can be found. After a lengthy and difficult calculation in which very rough approximations have to be made, Burgers succeeds in finding a lower limit 0·0007 for the resistance coefficient, which is of the same order of magnitude as the values found experimentally. In establishing this result use is made of Lorentz's elliptic vortex and the vortices introduced by Orr. It is interesting to note that the older empirical power laws for turbulent motion lead to a zero value for resistance as the Reynolds number becomes very large.

An approximate formula is also deduced for the distribution of the velocity of the mean motion over the breadth of the channel. This gives a velocity in the axis 1·25 times the mean velocity of flow; the experimental value is, however, about 1·1 times the mean velocity.

As the distribution obtained did not agree with the experimental values, Burgers attempted to penetrate further into the physical meaning of the dissipation condition. Since the mean motion was defined as a mean with respect to the time, a variation effected in the intensity of the relative motion, applied during a small interval, brought about a variation of the mean motion over the whole of the time considered. The latter variation then influenced the transmission of energy from the mean motion to the relative motion during the whole time considered.

In view of the unlikely nature of this explanation another method of attacking the problem was considered. The mean motion was defined as a mean with respect to x, and it was considered that in the normal state of turbulent motion as distinguished from tumultuous motion this method would probably lead to the same kind of result as the use of mean values with respect to time. It was hoped to get rid of certain difficulties of the first formulation in this way. By introducing these mean values with respect to x a new form for the dissipation condition was obtained, with a consequent change in the distribution function. A further long calculation involving the introduction of certain approximations and a special vortex field again led to results which were considerably higher than those found experimentally.

W. Tollmien (3) has undertaken the task of obtaining mathematically correct results by making use of Burgers' phase space. By using the forward and central difference quotients for the stream function, he deduced, in conjunction with other facts, that the velocity fluctuations with wave lengths of the order of magnitude of the lattice spacing ϵ preponderate to an unallowable degree. This shows that the introduction of the point lattice into the continuous fluid cannot be justified. Tollmien proposes to abandon the dissipation condition as a subsidiary condition governing the turbulence statistics, and the details of his method, whereby he proposes to overcome the essential difficulty of Burgers' hypothesis, have been published recently.

In 1933 Burgers published four papers in which a point of view suggested in his last paper of 1929 is elaborated. The completely turbulent motion of a fluid between two parallel walls is again considered. The axes are taken as before. The mean motion in the x-direction is denoted by U, which is a function of y and the time t. The components of the relative motion are denoted by u', v', and the z-component of the vorticity by ζ'. The Reynolds number R of the motion is given by $\rho v_0 h/\mu$. The stream function ψ for the relative motion is again introduced; with it u', v', ζ' are connected by the formulae:

$$u'=\frac{\partial\psi}{\partial y},\quad v'=-\frac{\partial\psi}{\partial x},\quad \zeta'=\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}=-\nabla^2\psi.$$

As the stream function ψ must satisfy the boundary conditions $\psi=0$, $\partial\psi/\partial y=0$ for both $y=-\frac{1}{2}$ and $y=+\frac{1}{2}$, ψ may be developed according to the normal solutions of a differential equation of the

fourth order. This equation is derived from the distribution function and will be discussed later. An N-dimensional phase space is now postulated with coordinates $\xi_1, \xi_2, \ldots \xi_n$. The coordinates in this space are given by the amplitudes or coefficients with which the various normal solutions occur in the development. These coordinates thus serve as a framework for the construction of cells in the phase space.

Burgers now considers only quantities defined for instantaneously observed modes of motion. The space between the two parallel walls is observed at M instants, separated by equal intervals of time, M being a large number. It is assumed that all the possible modes of motion of the system under consideration are classified. This classification must satisfy the condition that all the enumerated modes of motion are of equal weight. This hypothesis is introduced since there does not appear to be any method whereby the fields could be built up in various ways from other units. There is no simpler quantity which might be used to build up the values of ψ at the points in the phase space.

The sequence of M pictures obtained in the above way may then be regarded as defining a microscopically described state of the system. For convenience of discussion, the various modes of motion are numbered consecutively 1, 2, 3 ..., and the number of times, $n_1, n_2, n_3 \ldots$, each one of them occurs in the sequence of M pictures can be determined. In a statistical description it is evident that only the numbers are relevant, the order in which the various modes occur having no effect upon the mean values which will be determined. The set of numbers, $n_1, n_2, n_3 \ldots$, then uniquely determines a statistically described state. Corresponding to one ensemble of fields of flow for the relative motion, there may be effected $M!$ permutations among the M pictures. When those permutations in which modes of motion of the same type change place, or differ only slightly, are not counted, then there remain $W = M!/n_1!\, n_2!\, n_3! \ldots$ differently arranged sequences corresponding to the given set of numbers $n_1, n_2, n_3 \ldots$.

The conditions which must be satisfied by a statistically described state of the system will now be considered. To obtain the first, which may be termed the shearing force condition, the usual hydrodynamical non-dimensional equations containing the second-order terms are used. A mean value with respect to x for any function ϕ of the variables x, y, t is introduced by means of the equation $\overline{\phi} = L^{-1}\int \phi \, dx$ where the integration is over a great length L compared with the channel breadth h. Mean quantities are introduced into the two-dimensional equations, and terms small compared with the length L are neglected. The motion is again decomposed into the mean motion and the continually fluctuating relative motion. When the fluctuations are sufficiently numerous and distributed at random, certain rules which the mean quantities obey have been developed by various investigators. Bateman has given these rules in the form:

$$\overline{\overline{a}} = \overline{a}\,; \quad \overline{\overline{a}\omega} = \overline{a}\,.\,\overline{\omega}\,; \quad \overline{a+\omega} = \overline{a} + \overline{\omega}\,; \quad \overline{\frac{\partial a}{\partial s}} = \frac{\partial \overline{a}}{\partial s},$$

where s denotes any one of the variables x, y, t. These rules have been shewn to be valid to a sufficient degree of accuracy. Using these rules, and putting $2C$ for $-\overline{\partial p}/\partial x$, which can be shown to be constant, the first fundamental equation of the problem is deduced in the form :

$$\frac{\partial U}{\partial t}=2C-\frac{\partial}{\partial y}(\overline{u'v'})+R^{-1}\frac{\partial^2 U}{\partial y^2}. \quad \text{.....................(1)}$$

The dissipation condition is again used as a subsidiary relation which governs the states of the system. This condition may be obtained directly from the hydrodynamical equations or in a more simple way as given by Burgers. At any given instant the mean value, per unit length of the channel, of the energy put into the system is given by the pressure drop, multiplied by the total rate of flow. In non-dimensional notation this is $2C$. The rate of dissipation, on the other hand, is determined by the mean square of the vorticity of the real motion, integrated over the channel breadth. The equation for the rate of increase of the kinetic energy E of the flow per unit length of the channel is then :

$$\frac{dE}{dt}=2C-R^{-1}\int dy\left\{\overline{\zeta'^2}+\left(\frac{\partial U}{\partial y}\right)^2\right\}, \quad \text{..................(2)}$$

since the vorticity of the mean motion is $-\partial U/\partial y$.

The relations are now developed from which the distribution function is derived. If the motion shall be stationary, it appears that the mean value of $\partial U/\partial t$, taken with respect to the time, must eventually be zero, if the interval of time taken is long enough. If the right-hand side of (1) is indicated by σ_m for the mode of motion numbered m in the enumeration of possible modes of motion constituting a state of the system, the condition $\sum\limits_m n_m\sigma_m=0$ must then be satisfied for all values of y. We shall not, however, require that it is satisfied also for y exactly equal to $\pm\frac{1}{2}$. It is also obvious that the condition $\sum\limits_m n_m=M$ must be satisfied. The right-hand side of (2) is indicated by ϵ_m for the type of motion numbered m. Any state of the system must then satisfy the condition $\Sigma n_m\epsilon_m=0$, which applies to the whole field at once.

In order to avoid the use of the conception of a most probable distribution, Fowler's method of calculating averages by means of a partition function is used. Average values are thus taken in preference to most probable values to determine the properties of the system. An endeavour is now made to calculate the statistical mean values of the n_m taken over a large number of sequences, each consisting of M observations. The weight of a statistically described state is given by W, and so the statistical mean value of any one of n's is determined by $\overline{n}_m=\sum\limits_m{}^* Wn_m/\sum\limits_m{}^* W$, where the summation Σ^* is to be extended over all the values of the n's consistent with the relations :

$$\sum_m n_m = M \text{ ......(3a) ; } \sum_m n_m\sigma_m = 0 \text{ ......(3b) ; } \sum_m n_m\epsilon_m = 0. \text{ (3c)}$$

If (3*b*) is integrated over the channel, the relations (3*b*) and (3*c*) can be combined in the form : $\Sigma n_m K_m = 0$ (3*a*), where $K_m = \epsilon_m - \int dy \lambda \sigma_m$, and λ is an arbitrary function of y. By introducing the partition function $\phi(z) = \sum_m z^{K_m}$, and using the method of steepest descents, the statistical mean values are obtained in the form :

$$\overline{n}_m = \exp(c + \beta K_m)$$

where c and β are numerical constants. Substituting the values from equations (1) and (2) in this result and integrating by parts, we obtain the result :

$$\overline{n}_m = \exp\left(c + \beta \int dy \left[R^{-1} \frac{\partial U}{\partial y} \frac{d\lambda}{dy} - R^{-1} \left(\frac{\partial U}{\partial y}\right)^2 - \overline{u'v'} \frac{d\lambda}{dy} - R^{-1} \overline{\zeta'^2} \right]_m \right), \text{(4)}$$

where the index m indicates that the quantity between the square brackets must be calculated for the mode of motion numbered m. The pressure drop has been eliminated from (4) by setting $\int \lambda \, dy = 1$. The constants c and β are to be determined by the condition that (3*a*) and (4) must be satisfied ; the condition for λ is that (3*b*) shall be fulfilled for all values of y between $+\frac{1}{2}$ and $-\frac{1}{2}$. Since (3*b*) need not be satisfied for $y = \pm\frac{1}{2}$, we assume that λ is zero there.

Since the exponent in (4) can be divided into two distinct parts, one depending on the mean motion and the other on the relative motion, we may for the moment consider the distribution of the relative motion by discussing the formula :

$$\overline{n}_m = \exp\left(\beta \int dy \left[-\overline{u'v'} \frac{d\lambda}{dy} - R^{-1} \overline{\zeta'^2} \right]_m \right). \text{ ....(4a)}$$

Burgers makes further progress with the theory by basing the method of enumerating the various modes of motion upon a special hypothesis. Assuming that some particular value of λ has been chosen, we then investigate which stream functions ψ give a maximum value to the fraction :

$$-\int dy \, \overline{u'v'} \frac{d\lambda}{dy} \Big/ R^{-1} \int dy \, \overline{\zeta'^2}. \text{ ....(5)}$$

For this purpose we put $\psi = \chi_{\text{I}}(y) \cos \alpha x + \chi_{\text{II}}(y) \sin \alpha x$, which function occurs in the theoretical researches on the stability of laminar motion. Afterwards this result is generalised to include all possible combinations of stream functions. By finding expressions for $-\overline{u'v'}$ and $\overline{\zeta'^2}$, and denoting differentiation with respect to y by dashes on the χ functions, the fraction (5) becomes :

$$\beta \int dy \left[\frac{\alpha}{2} (\chi_{\text{I}}' \chi_{\text{II}} - \chi_{\text{I}} \chi_{\text{II}}') \frac{d\lambda}{dy} \Big/ \tfrac{1}{2} R^{-1} \{ (\chi_{\text{I}}'' - \alpha^2 \chi_{\text{I}})^2 + (\chi_{\text{II}}'' - \alpha^2 \chi_{\text{II}})^2 \} \right]_m . \text{ (5a)}$$

Using the well-known rules for the variational problem for the two functions χ_{I}, χ_{II} (which can be varied independently of each other), we are led to the system of equations :

$$\chi_{\mathrm{I}}^{\mathrm{IV}} - 2\alpha^2\chi_{\mathrm{I}}'' + \alpha^4\chi_{\mathrm{I}} + \alpha R\Omega\left(\frac{d\lambda}{dy}\chi_{\mathrm{II}}' + \tfrac{1}{2}\frac{d^2\lambda}{dy^2}\chi_{\mathrm{II}}\right) = 0,$$

$$\chi_{\mathrm{II}}^{\mathrm{IV}} - 2\alpha^2\chi_{\mathrm{II}}'' + \alpha^4\chi_{\mathrm{II}} - \alpha R\Omega\left(\frac{d\lambda}{dy}\chi_{\mathrm{I}}' + \tfrac{1}{2}\frac{d^2\lambda}{dy^2}\chi_{\mathrm{I}}\right) = 0,$$

which may be combined into the single one :

$$\chi^{\mathrm{IV}} - 2\alpha^4\chi'' + \alpha^2\chi - i\alpha R\Omega\left(\frac{d\lambda}{dy}\chi' + \tfrac{1}{2}\frac{d^2\lambda}{dy^2}\chi\right) = 0, \quad \text{.........(6)}$$

where $\chi = \chi_{\mathrm{I}} + i\chi_{\mathrm{II}}$. To obtain these results repeated integration by parts and the boundary conditions

$$\chi_{\mathrm{I}} = \chi_{\mathrm{II}} = \chi_{\mathrm{I}}' = \chi_{\mathrm{II}}' = 0 \text{ at } y = \pm\tfrac{1}{2}$$

are used. The equation satisfied by the conjugate function

$$\bar{\chi} = \chi_{\mathrm{I}} - i\chi_{\mathrm{II}}$$

is obtained by changing i into $-i$. The number Ω in (6) is the characteristic number; solutions satisfying the above boundary conditions can be obtained only for a definite series of values of this parameter. If the characteristic values are denoted by Ω_k, the corresponding normal solutions are written χ_k. Burgers develops certain properties of the solutions of (6), which hold for two functions χ_k, χ_l belonging to different characteristic numbers Ω_k, Ω_l. The normalising conditions assumed make the characteristic values of the parameter Ω all positive, provided α is positive.

In order to retain the restriction of α to positive values, and for convenience of notation, Burgers introduces the complete stream function ψ_m for the mode of relative motion numbered m in the form

$$\psi_m = \tfrac{1}{2}\sum_{ak}\{e^{-i\alpha x}(A_{ak}\chi_{ak} + B_{ak}\bar{\chi}_{ak}) + e^{i\alpha x}(\bar{A}_{ak}\bar{\chi}_{ak} + \bar{B}_{ak}\chi_{ak})\}, \quad \text{...(7)}$$

where $\quad A_{ak} = A^{\mathrm{I}}{}_{ak} + iA^{\mathrm{II}}{}_{ak}, \quad \bar{A}_{ak} = A^{\mathrm{I}}{}_{ak} - iA^{\mathrm{II}}{}_{ak}$, etc.

Any mode of relative motion is now specified by the values of the A's and B's, and in calculating statistical mean values the summation with respect to the number m (*i.e.* the summation over the phase space) can be replaced by an integration over the A's and B's. In (7), the normal functions are denoted by χ_{ak}, in order to denote their dependence on the parameter α.

From (7), expressions for $-\overline{u'v'}$ and $\overline{\zeta'^2}$ are easily deduced. The following expressions for the integrals are then obtained by making use of the properties of the solution of equation (6) :

$$-\int dy\,\overline{u'v'}\,\frac{d\lambda}{dy} = \frac{1}{2R}\sum_{ak}\alpha^3(A_{ak}\bar{A}_{ak} - B_{ak}\bar{B}_{ak}), \quad \text{.........(8a)}$$

$$\int dy\,\overline{\zeta'^2} = \tfrac{1}{2}\sum_{ak}\alpha^3\Omega_{ak}(A_{ak}\bar{A}_{ak} + B_{ak}\bar{B}_{ak}). \quad \text{.........(8b)}$$

Consequently the distribution function (4*a*) takes the form :

$$\bar{n}_m = \exp\left(\tfrac{1}{2}\beta R^{-1}\sum_{ak}\alpha^3\{A_{ak}\bar{A}_{ak}(\Omega_{ak} - 1) + B_{ak}\bar{B}_{ak}(\Omega_{ak} + 1)\}\right). \quad \text{...(9)}$$

This distribution function must remain finite for all values of the variables. To satisfy this condition the numbers Ω_{ak} must be greater than unity. The statistical mean values $\overline{f}$ of any quantity f depending on the A's and B's can then be calculated by means of the formula:

$$\overline{f}=(\sum_m \overline{n}_m f_m)/(\sum_m \overline{n}_m),$$

where the summation with respect to m may be replaced by an integration over the A's and B's.

Further development of the theory depends on the solution of the differential equation (6) of the fourth order. Burgers now considers that in some region in the neighbourhood of a wall the relative motions for various distances from the wall may be considered as similar, the scale being proportional to the distance. As the scale in the direction of x is determined by the parameter α, it may be asked if, for any given value of the number k, the functions χ_{ak} might be functions of a single variable $\xi=\alpha\eta$, where $\eta=y+\frac{1}{2}$. As the presence of the wall at $y=+\frac{1}{2}$ disturbs the similarity, it must be assumed that this wall is situated at a great distance, so that, if it may happen that functions χ_{ak} are found which decrease sufficiently quickly for large values of ξ, the presence of this second wall may be without appreciable influence upon them.

A plausible form for λ is now assumed. It is seen that α and η disappear as separate variables from the equation (6), if the condition, $\dfrac{d\lambda}{d\eta}=\dfrac{b}{R\eta^2}$, where b is a positive constant, is applied in the neighbourhood of the wall $\eta=0$. But in order not to violate the condition $\lambda=0$ for $\eta=0$, the condition $\dfrac{d\lambda}{d\eta}=\dfrac{b}{R\delta^2}$, where δ is a small constant, must be assumed for the region $\eta<\delta$. A different treatment is thus introduced for a boundary layer near the wall $\eta=0$. The second condition leads to an equation which may be solved by means of functions of the type $\exp(m\xi)$, where m is one of the four roots of an equation of the fourth degree. The first condition leads to an equation which when transformed by Laplace's method becomes a hypergeometric equation. After a lengthy calculation a system of normal functions, defined by the formula,

$$\chi_{ak}=N_k\alpha\eta\int d\zeta e^{a\eta\zeta}w_k(\zeta),$$

and integrated over a certain contour, is obtained. N_k is a numerical constant, while w_k is given by the formula:

$$w_k(\zeta)=t^{\alpha_1}(1-t)^3F(1-2k,\ 2+2i\sqrt{k^2+k},\ 4,\ 1-t),$$

where $t=(\zeta+1)/(\zeta-1)$, $\alpha_1=-k-1+i\sqrt{k^2+k}$ and F is the symbol for the hypergeometric series. By making use of the boundary conditions it is also found that the characteristic values of the parameter Ω_k are given by

$$b\Omega_k=4(2k+1)\sqrt{k^2+k}, \quad \text{.......................(10)}$$

where k is a positive integer. The case $k=0$ must be excluded, as Ω_k must be greater than zero.

So far it has been shown that for a special choice of the function λ a series of functions χ_{ak} can be developed, which allow the development of the stream function ψ for the relative motion in the form (7). At the same time the exponent of the distribution function (9) is transformed into a homogeneous function of the A's and B's. Apart from difficulties of mathematical technique it is then possible to calculate the statistical mean values of quantities of the type $\overline{A_{ak}A_{al}}$, etc. Further it would then be possible to write down expressions for $-\overline{u'v'}$ and for $\overline{\zeta'^2}$ as functions of η (or y), and so obtain values for the integrals (8a) and (8b).

Before proceeding with these calculations, however, the problem presented by the determination of $\overline{U}$ will be considered. The general statistical principles set forth in the previous paragraphs are also intended to lead to the distribution of the mean motion. The point of view taken is that the statistical considerations should apply to the actual motion, built up of the sum of the relative and mean motions. If the actual motion is derived from a stream function Ψ a new set of coordinates must be used to describe the distribution of the mean values $\overline{\Psi}$ as a function of y. These coordinates constitute the η-space. The total phase space is called the ξ-η space. Now the exponent of the distribution function (9) clearly splits up into two parts, one part depending exclusively on the mean motion, the other part depending exclusively on the relative motion. The second group of coordinates can then be considered as orthogonal to the first. An investigation is now made of the function:

$$\exp\left(\beta R^{-1}\int\left\{\frac{\partial U}{\partial y}\frac{d\lambda}{dy}-\left(\frac{\partial U}{\partial y}\right)^2\right\}dy\right), \quad \ldots\ldots\ldots\ldots\ldots(11)$$

which depends on the mean motion. A certain difficulty presents itself in that the spacing of the coordinates which describes the course of the function Ψ should bear a relation to the spacing of the coordinates used to describe ψ. Again the exponent of (11) is not homogeneous with respect to U, and so does not lead to a variational problem from which a system of normal functions can be deduced.

Burgers tries whether a simple Fourier expansion of the type

$$\overline{\Psi}=\Sigma U_n \sin(2n+1)\pi y$$

may be used. This form ensures that $\partial\overline{\Psi}/\partial y$ shall be zero at the walls of the channel. By introducing this expansion and using the boundary conditions for $\overline{\Psi}$ and λ, and also the condition $\int\lambda\, dy=1$ into (11), the statistical problem can be worked out. The result is

$$\overline{U}_n=\frac{\lambda_n}{2\pi(2n+1)}+\frac{24}{\pi^4}\frac{(-1)^n}{(2n+1)^4},$$

where $\lambda=\Sigma\lambda_n \cos(2n+1)\pi y$. This leads to the formula:

$$\frac{\partial\overline{U}}{\partial y}=\tfrac{1}{2}\frac{d\lambda}{dy}-6y. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

This formula unfortunately leads to values which in the central part of the channel are too high.

Notwithstanding this result it is interesting to see how matters stand with regard to the application of equation (3*b*). Integrating this equation with respect to y, we obtain :

$$2Cy - \overline{u'v'} + R^{-1}\partial\overline{U}/\partial y = 0. \quad \text{.....................(13)}$$

The course assumed for $d\lambda/dy$ and the condition $\int \lambda\, dy = 1$ worked out lead to the approximate relation $\delta = 2b/R$, so that there is only one adaptable parameter in λ. Consequently, as the pressure drop $2C$ is still unknown, we can make (13) fit at two points at most to obtain relations between b and C. Taking as one of these points $\eta = 0$, and the other at a distance from the wall great compared with δ, Burgers obtains the approximate relations :

$$C \cong 1/8b, \quad \text{and} \quad C \cong -\overline{u'v'} \text{ for } (\tfrac{1}{2} > \eta > \delta).$$

The third relation between the three constants must be obtained from the dissipation condition (3*c*). Written explicitly it takes the form :

$$C - \frac{1}{R}\int_0^{\frac{1}{2}} d\eta \overline{\left(\frac{\partial U}{\partial y}\right)^2} - \frac{1}{R}\int_0^{\frac{1}{2}} d\eta\, \overline{\zeta'^2} = 0.$$

A very complicated approximate calculation leads to the relation :

$$\frac{1}{R}\int_0^{\frac{1}{2}} d\eta\, \overline{\zeta'^2} \cong \frac{2}{\beta\theta}\int d\alpha \Sigma \frac{\Omega_k^{\,2}}{\Omega_k^{\,2} - 1}, \quad \text{.................(14)}$$

which appears to be divergent both with respect to k and with respect to α. By using the asymptotic expansion for w_k, it appears probable that k cannot increase beyond a certain amount owing to the presence of the second wall. The total dissipation thus threatens to become infinite unless a limit can be found to the system of modes of motion. A discussion of the properties of the system of normal functions determined by equation (6) with an arbitrary form of $d\lambda/dy$ will be necessary to settle this point.

Again, the relation $C \cong 1/8b$ and equation (10) lead to the result, $C > 0{\cdot}0074$, which is many times greater than the experimental values.

In his concluding paper Burgers discusses several points in the theory which are open to criticism. There is no doubt that the restriction to two dimensions must have a considerable effect on the results obtained. Again the connection between $d\overline{U}/dy$ and $-\overline{u'v'}$ given by equation (13) appears to be of doubtful validity. In the central part of the channel this connection is too weak to have an appreciable influence upon the course of $d\overline{U}/dy$, which may be effected by some other mechanism. It may be that some other conditions must still be introduced.

The form in which the statistical method has been used can only be justified by its apparent simplicity, and is modelled upon the statistical treatment of a conservative system. It has been seen that in the case of a system having an infinite number of degrees of freedom the method leads to a divergence in the dissipation. A solution of this difficulty may be found by considering in more detail the dissipative nature of the system. This is closely connected with the principles involved in Liouville's theorem.

Burgers has succeeded in constructing a simpler system of equations:

$$\frac{\partial u}{\partial t} = F - \kappa v^2 + \mu \frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial v}{\partial t} = \kappa u v + \mu \frac{\partial^2 v}{\partial y^2},$$

where u and v are functions of y and t, to which the same formal treatment can be applied as with the hydrodynamical equations. By considering the variation with respect to time of a group of points moving through the phase space introduced for this system, it appears likely that a means of restricting the number of degrees of freedom to be considered in the statistics of turbulent motion may be obtained; or perhaps of the introduction of a certain weight function which will eventually make the expression for the total dissipation convergent. G. F. P. T.

References:

(1) L. Prandtl, *Journ. Germ. Eng. Soc.*, 77, 105, 1933.

(2) J. M. Burgers, *Proc. Acad. Sciences, Amsterdam*, 32, pp. 414, 643, 818, 1929; 36, pp. 276, 390, 487, 620, 1933.

(3) W. Tollmien, *Zeit. f. ang. Math. u. Mech.*, 13, 331, 1933.

---

992. Euclid as a Christian Name.

I have come across three instances:

1. Euclid Speidell, who flourished in the second half of the seventeenth century, teaching mathematics " next door to the Cock ", Bow Street, Covent Garden. He published, in 1688, " Logarithmotechnia: or the making of numbers called logarithms, to 25 places, from a geometrical figure, with speed, ease, and certainty." He was also responsible for a corrected edition of Henry Gellibrand's *Epitome of Navigation* (1693). His father, John Speidell, was an early follower of Napier, his " New Logarithmes " appearing in 1619.

2. Euclid Baker (d. 1714), Master Gunner at the Castle and Chief Clerk of the Fortifications at Bombay. For him see Burnell: *Bombay in the days of Queen Anne* (Hakluyt Society, 1933), p. 23, note 1. He had a brother named Copernicus.

3. Euclid Curry Butterworth took his M.A. degree at the University of Cambridge from Christ's College in 1932. [Per Mr. F. P. White.]

993. There are Japanese-made clothes on sale in London at prices 100 per cent. or more under the nearest British-made equivalent.—*Morning Post*, Dec. 1, 1933, p. 13, col. 2. [Per Mr. I. FitzRoy Jones.]

EXTENSIONS AND IMPLICATIONS OF SIMSON'S LINE.

BY N. M. GIBBINS.

1. LET ABC be a triangle, H its orthocentre, P a point on the circumcircle, and D, E, F the feet of the perpendiculars from P on BC, CA, AB respectively. Then it is known that D, E, F are on a straight line which bisects PH. Through P draw lines making the angle $\frac{1}{2}\pi - \alpha$ in the same sense with PD, PE, PF, PH to meet the sides in D', E', F' and a line through H perpendicular to PH in H'. Then D', E', F' are also on a line which bisects PH'. Call this line the Simson line (α) of P with respect to ABC.

2. Let O be the circumcentre of ABC and construct the triangle $OO'H$ similar in the same sense to $PH'H$. Let N', T be the middle points of OO', PH'. Then the triangles HOP, $HO'H'$, $HN'T$ are similar, whence $N'T = \frac{1}{2}OP \operatorname{cosec}\alpha$. Hence for a given value of α the locus of T as P describes the circumcircle is a fixed circle. We may call it the nine-points circle (α) of the triangle ABC, for it obviously passes through nine points which can be constructed (as in § 1) from the special points of the nine-points circle itself.

If N is the nine-points centre and I the incentre, and OII' be drawn similar in the same sense to ONN', then the triangles $OI'N'$ and OIN are similar, so that $I'N' = IN \operatorname{cosec}\alpha$. Now $IN = \frac{1}{2}R - r$; thus $I'N' = \frac{1}{2}R \operatorname{cosec}\alpha - r \operatorname{cosec}\alpha$. Hence if a circle be drawn with centre I' and radius $r \operatorname{cosec}\alpha$, the nine-points circle (α) will touch it. Similarly it will touch the circles whose centres I_1', I_2', I_3' are constructed, as in the case of I and I', from I_1, I_2, I_3 and whose radii are $r_1 \operatorname{cosec}\alpha$, $r_2 \operatorname{cosec}\alpha$, $r_3 \operatorname{cosec}\alpha$ respectively.

3. As α varies each of these sets of circles has an envelope. Let S be a point, SC a given length and let SCg be constructed with $\angle SCg = \frac{1}{2}\pi$ and $\angle SgC = \alpha$. Produce SC to S' so that $CS' = SC$. Let the circle gSS' meet the circle whose centre is g and radius $\rho \operatorname{cosec}\alpha$ in P and P'. Then, by Ptolemy's theorem,

$$Pg \,.\, SS' = (SP + S'P) \,.\, Sg, \quad \text{or} \quad SP + S'P = 2\rho.$$

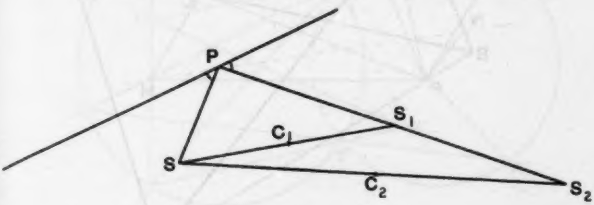


FIG. 1.

Hence Pg is the normal at P to the conic which has S as one focus, C as centre, and 2ρ as the length of its major axis; so that the envelope of the circles is this conic, all the circles having double contact with it.

4. If two conics have one focus S in common and touch a given line at the same point, then the distance between their centres is equal to the sum or difference of their major semi-axes, and conversely. When the other foci S_1 and S_2 are both on the same side of the line as S the conics are both ellipses; when S_1 and S_2 are on the opposite side of the line to S they are both hyperbolas. In these cases we have the difference. When S_1 and S_2 are on opposite sides of the tangent, one conic is an ellipse and the other a hyperbola and we have the sum.

The envelope corresponding to the nine-points circles (α) is the conic having O and H as foci and major axis equal to the circumradius. Since $IN=\frac{1}{2}R-r$, etc., this conic touches the four conics having O as one focus, the incentre and excentres as centres, and the radii of the incircle and excircles as major semi-axes respectively.

When the nine-points envelope is an ellipse, so is the incentre envelope and the others are hyperbolas; when the nine-points envelope is a hyperbola so is the incentre envelope and the others are ellipses. The criterion is whether OH is less than or greater than R, that is, whether the triangle is acute-angled or obtuse-angled.

When $OH=R$, that is, when the triangle is right-angled, the nine-points envelope degenerates into the points O and H, so that the other four envelopes all pass through H—the right-angled vertex of the triangle.

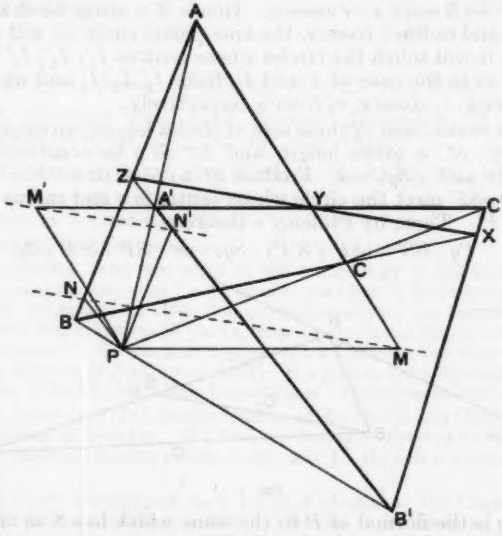


FIG. 2.

5. Beginning with any transversal XYZ (instead of the special transversal $D'E'F'$), draw lines through X, Y, Z making with

BC, CA, AB respectively angles α in the same sense, and forming a similar triangle $A'B'C'$. Then AA', BB', CC' meet in a point P. Now $AYA'Z$ is cyclic, hence $\angle YAA' = \angle YZA'$; also $BZXB'$ is cyclic, hence $\angle XZB' = \angle XBB'$. Hence $\angle CAP = \angle CBP$, and therefore $ABPC$ is cyclic; similarly $A'PB'C'$ is cyclic, so that P is on the circumcircles of ABC and $A'B'C'$. Also the tangent at P to the circle ABC makes with PB the angle PCB in the alternate segment; and the tangent at P to the circle $A'B'C'$ makes with PB' the angle $PC'B'$ in the alternate segment. But the sum of the angles PCB, $PC'B'$ is α, so that the circumcircles of ABC and $A'B'C'$ intersect at the angle α.

6. In the figure draw PM parallel to $A'C'$ to meet AC in M, and PN parallel to $A'B'$ to meet AB in N. Then MN is the Simson line (α) of P with respect to ABC. $PMAN$ is cyclic, hence

$$\angle AMN = \angle APN = \angle AA'Z = \angle AYZ.$$

Hence MN is parallel to XYZ. Again, draw PM' parallel to AC to meet $A'C'$ in M', and PN' parallel to AB to meet $A'B'$ in N'. Then $M'N'$ is the Simson line $(\pi - \alpha)$ of P with respect to $A'B'C'$, and similarly it is parallel to XYZ. Since $PMYM'$ is a parallelogram, MM' and PY bisect one another. Hence the Simson line (α) of P with respect to ABC and the Simson line $(\pi - \alpha)$ of P with respect to $A'B'C'$ are equidistant from P and XYZ. Hence if Q is a point such that the former line bisects PQ, and Q' a point such that the latter line bisects PQ', it follows from proportionate division that the transversal bisects QQ'. Thus the transversal bisects the join of the "displaced" orthocentres of the triangles, it being noted that the "displacements" must be in opposite directions. When $\alpha = \frac{1}{2}\pi$, the circumcircles of ABC and $A'B'C'$ cut orthogonally, and the transversal bisects the join of their orthocentres.

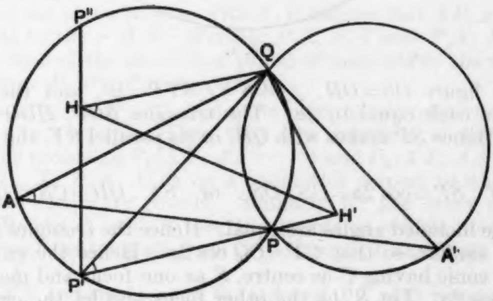


FIG. 3.

7. For any transversal parallel to the Simson line (α) of P with respect to ABC, P is the centre of perspective of ABC and every triangle $A_r'B_r'C_r'$. Also the sides of the triangles $A_r'B_r'C_r'$ are respectively parallel to one another. Hence the loci of points

similarly placed in the triangles (in particular the orthocentres) are straight lines through P.

If H and H' are the orthocentres of the triangles ABC and $A'B'C'$ and Q the other point of intersection of the circumcircles, the triangles HQH' and AQA' are similar. Hence (Fig. 3)

$$\angle QHH' = \angle QAP = \angle QP'H'.$$

Hence $HP'H'Q$ is cyclic. But $\angle HQH' = \angle AQA' = \pi - \alpha$. Hence $\angle HP'H' = \alpha$. Hence if PH_1' be drawn making the angle $\frac{1}{2}\pi - \alpha$ with PH', it is perpendicular to HP', that is, to HP'' where PP'' subtends the angle 2α at O. This result enables us to find the envelope of PH_1' as P moves round the circle ABC.

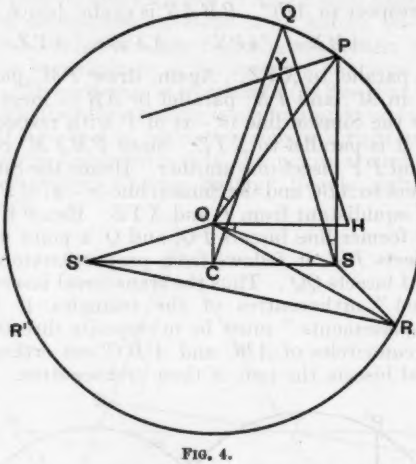


FIG. 4.

In the figure $OS = OH$, $\angle OCS = \angle SYP = \frac{1}{2}\pi$, and the marked angles are each equal to 2α. The triangles SOP, HOQ are congruent. Hence SP makes with QH, or its parallel SY, the angle 2α. Hence

$$SY : SP = \cos 2\alpha = CS : OS, \quad \text{or} \quad SY : QH = CS : OH.$$

Also the included angles are equal. Hence the triangles CYS and OQH are similar, so that $CY = OQ \cos 2\alpha$. Hence the envelope of PH_1' is a conic having C as centre, S as one focus and major semi-axis $OP \cos 2\alpha$. Let S' be the other focus and let the circle $S'OS$ meet the circle ABC in R and R'. Then by Ptolemy's theorem $SR + S'R = 2\,OR \cos 2\alpha$, while OR bisects the angle $S'RS$. Hence R is a point on the envelope and OR the normal thereat. Hence the envelope has double contact with the circumcircle of ABC for all values of α. N. M. GIBBINS.

GENERALISATION OF PLÜCKER'S THEOREM.

By N. M. Gibbins.

1. It is known that if S be one of a range of conics touching four given lines, and if XP be a variable line through a given point X, then (a) there exists a line p' which is conjugate to XP with respect to every S; and (b) p' touches a fixed conic σ, which is the eleven-line conic of X with respect to the range, and which is also the envelope of the polar of X with respect to the range. Let us call this conic the σ of X.

It is clear from (a) that as X moves along a fixed line l, the σ's of X all touch l', the locus of the poles of l with respect to the range.

2. Let X and Y be two fixed points in the plane of the range. Draw the two tangents from Y to the σ of X, and through X draw XP_1 conjugate to one tangent with respect to one conic of the range (and therefore with respect to every conic of the range) to meet it in P_1; and similarly draw XP_2 to meet the other tangent in P_2.

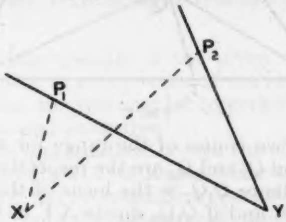


Fig. 1.

Then P_1 and P_2 are the two points (and there are only two points) for which P_1X, P_1Y and P_2X, P_2Y are conjugate lines for every conic of the range.

Beginning the above process with Y, it follows that XP_1 and XP_2 are tangents to the σ of Y. Further P_1X, P_1Y and P_2X, P_2Y are the double lines of the involution pencil of tangents to the conic of the range from P_1 and P_2 respectively.

3. Let A, A'; B, B'; C, C' be the pairs of opposite vertices of the given quadrilateral. Then they are members of the range. Hence by the preceding $P_1(XY, AA') = -1$ and $P_2(XY, AA') = -1$. Hence P_1, P_2, X, Y, A, A' lie on a conic with respect to which XY and AA' are conjugate chords. Similarly for the other pairs of vertices. Hence P_1 and P_2 are the points at which these fixed conics intersect the tangents from X to the σ of Y and those from Y to the σ of X.

Further XY and the locus of the poles of XY with respect to the range harmonically separate A, A'; B, B'; C, C'.

4. The σ of X and the σ of Y are also the envelopes of the polars of X and Y respectively with respect to the conics of the range. Hence there is one conic of the range with respect to which P_1Y is the polar of X, and hence with respect to that conic the polar of Y

passes through X. But the polar of Y is one of the tangents to the σ of Y and it is not XP_1; hence it is XP_2. Let P_1Y and XP_2 meet in Q_1. Then there is one conic of the range with respect to which Q_1XY is a self-conjugate triangle. Similarly, if P_2Y and XP_1 meet in Q_2, there is another conic of the range with respect to which Q_2XY is a self-conjugate triangle.

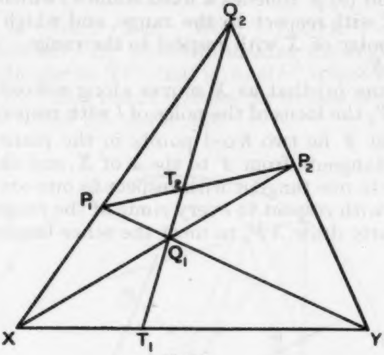


FIG. 2.

Hence there are two conics of the range for which X and Y are conjugate points, and Q_1 and Q_2 are the respective poles of XY with respect to them. Hence Q_1Q_2 is the locus of the poles of XY with respect to the range, and if Q_1Q_2 meets XY in T_1 and P_1P_2 in T_2, then T_1 and T_2 are the points of contact of those conics of the range which touch XY and P_1P_2 respectively.

Also, by the harmonic property of the quadrilateral

$$(T_1T_2,\ Q_1Q_2) = -1.$$

5. It is known that if X and Y are two points and S a conic, the locus of the point of intersection, P, of lines PX and PY conjugate with respect to S is a conic passing through X and Y. Call this conic the Σ of S.

If Q is the pole of XY with respect to S, XQ and XY are conjugate lines with respect to S, as are also YQ and YX. Hence as P tends to X and to Y, QX and QY are the tangents to Σ at X and Y; that is, Σ and S have in common the pole of XY. Other special points on Σ are the points of contact of the tangents to Σ from X and Y.

If S touches XY, one part of the locus is XY, and, by the preceding paragraph, the second part is the join of the other points of contact of the tangents to S from X and Y. If X and Y are conjugate points with respect to S, the pole of XY with respect to S is the only real point on Σ.

Let now S be one of a range of conics. There are two points P_1 and P_2 for which P_1X, P_1Y and P_2X, P_2Y are pairs of conjugate lines with respect to every S. Hence the Σ of every S passes through P_1 and P_2 as well as through X and Y.

The Σ of that conic which touches XY consists of the lines XY and P_1P_2. The Σ's of the two conics with respect to which X and Y are conjugate points are Q_1 and Q_2. Also the locus of the poles of XY with respect to the conics Σ is the same as for the conics S, namely, Q_1Q_2.

6. Projecting X and Y to ω and ω', the circular points at infinity, the results of § 5 become Plücker's theorem, namely: the director circles of the conics of a range form a coaxal system of which the radical axis is the directrix of the parabola of the range, and the limiting points are the centres of the rectangular hyperbolas of the range. This is consistent with § 4, where now the projection of T_2 is the middle point of the join of the projections of Q_1 and Q_2. It is also consistent with the last paragraph of § 5, since the line of centres of the circles is the join of the limiting points.

Three conics of the range are the pairs of opposite vertices of the given quadrilaterals, and their director circles are the circles having these pairs of opposite vertices as extremities of a diameter. This is consistent with § 3.

7. The diagonal-line triangle of the given quadrilateral is self-conjugate with respect to every conic of the range; and therefore, by a known theorem, the rectangular hyperbolas of the range pass through its incentre and excentres.

Hence if the given quadrilateral is such that its sides pass through the incentre and excentres of its diagonal line triangle, these are the points of contact of the rectangular hyperbolas of the range with the given lines, and hence the curves must coincide. In that case the directrix of the parabola of the range passes through their coincident centres and the director circles of the conics of the range touch there, the directrix of the parabola being the common tangent.

8. Since a diameter of a circle cuts harmonically any circle orthogonal to it, we have, from the last paragraph of § 6, that the orthogonal set of coaxal circles consists of those circles with respect to which the pair of opposite vertices of the given quadrilateral are conjugate points. Among these are the self-polar circles of the triangles formed by taking three out of the four lines, and also the circumcircle of the diagonal line triangle of the quadrilateral. Further their line of centres is the radical axis of the director circles, that is, the directrix of the parabola of the range, while they all pass through the limiting points of the former set, that is, through the centres of the rectangular hyperbolas of the range.

Now the centres of the self-polar circles referred to above are always real, being the orthocentres of the triangles, and if the centres of the two rectangular hyperbolas are real, these circles pass through two real points and are therefore themselves real. Hence the necessary and sufficient condition that the centres of the hyperbolas should be real, that is, that the system of director circles should have real limiting points, is that the four triangles formed by taking three out of the four given lines should all be obtuse-angled.

9. From § 6 we deduce the following theorems :

(*a*) The director circles of all conics touching three lines are orthogonal to the self-polar circle of the triangle formed by the lines.

(*b*) The director circles of all conics having a common self-conjugate triangle are orthogonal to the circumcircle of the triangle.

(*c*) The directrix of a parabola passes through the orthocentre of the triangle formed by three tangents, and through the circumcentre of any triangle self-conjugate with respect to it.

(*d*) The centre of a rectangular hyperbola lies on the self-polar circle of the triangle formed by any three tangents, and on the circumcircle of any triangle self-conjugate with respect to it.

(*e*) The orthocentres of the triangles formed by taking three out of four lines are collinear, by (*c*) above.

10. §§ 7 and 8 refer to the orthocentre, self-polar circle, and the incentre and excentres of a triangle. Hence before proceeding to generalise the implications of Plücker's theorem it is necessary to find the projections of these features of a triangle.

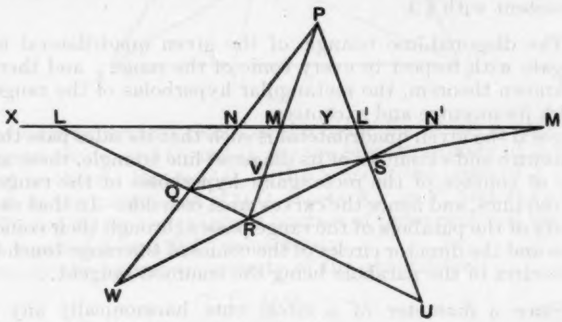


FIG. 3.

(*a*) Let *PQRS* be a quadrangle and *UVW* its diagonal-point triangle. Let any line cut the pairs of opposite sides in L, L'; M, M'; N, N', as in the diagram. Let X and Y be the double points of this involution range. Then

$$(XY, LL') = (XY, MM') = (XY, NN') = -1.$$

Also with respect to any conic through P, Q, R, S, X and Y are conjugate points and UVW a self-conjugate triangle. Let us call any one of the points P, Q, R, S the ***harmocentre*** of the triangle whose vertices are the other three. Projecting X and Y to ω and ω', we obtain four triangles and their orthocentres, while U, V, W project into the feet of the perpendiculars from the vertices on to the opposite sides.

(*b*) Consider the conic through X and Y with respect to which PQR is a self-conjugate triangle. Since $(XY, NN') = -1$, the polar of N passes through N'; and since PQ is the polar of R, the polar of N passes through R. Hence the polar of N is $N'R$, and this passes

through S. Hence the polar of S passes through N and similarly through M and L. Hence the polar of S is LMN or XY. Similarly we may prove that the conics through X and Y, with respect to which any one of the triangles whose vertices are three out of the four points P, Q, R, S is a self-conjugate triangle, have the fourth point as the pole of XY. Projecting as before, we have that the centre of the self-polar circle of a triangle is the orthocentre.

(c) Consider the conics having double contact at X and Y, S being the common pole. The tangents from U form an involution pencil of which UX and UY are a pair of conjugate rays and US a double ray; and since $(XY, LL') = -1$, UR is the other double ray. Also, by the harmonic property of the quadrangle, $U(WV, RS) = -1$; hence UW and UV are a pair of conjugate rays. Hence that conic which touches UW touches UV, and similarly touches VW.

Hence S is the pole of XY with respect to one of the conics through X and Y touching the sides of UVW; and similarly P, Q, R are the poles of XY with respect to the other three conics through X and Y touching the sides of UVW. Projecting, P, Q, R, S become the incentre and excentres of the projection of UVW; hence a rectangular hyperbola passes through the incentres and excentres of any triangle which is self-conjugate to it.

11. Projecting the results of § 8, the orthogonal circles become the conics Σ' (say), which pass through X and Y and have A, A'; B, B' as pairs of conjugate points, and hence also C, C', by Hesse's theorem. These conics all pass through Q_1 and Q_2 in the figure of § 4, and further every Σ' is harmotomic to every Σ.

The conics Σ' include the four conics through X and Y with respect to which three out of the four sides of the given quadrilateral are self-conjugate triangles, and also the conic through X and Y which circumscribes the diagonal-line triangle of the quadrilateral. Also the locus of the poles of XY with respect to the conics Σ' is P_1P_2.

Direct proofs of these theorems, regarding the conics Σ as conics through P_1P_2XY, and the conics Σ' as conics through Q_1Q_2XY, are implied in J. J. Milne's *Cross-Ratio Geometry*, Arts. 202-205.

12. We may also generalise the result of § 7. In § 10 (c), if U, V, W, X, Y are given, the points P, Q, R, S are constructed by finding rays URQ and USP such that

$$U(WV, RS) = -1 \quad \text{and} \quad U(XY, RS) = -1;$$

and similarly rays WRS and WQP such that

$$W(UV, RQ) = -1 \quad \text{and} \quad W(XY, RQ) = -1.$$

If then the sides of the original quadrilateral of the range pass, one each, through the four points constructed as above from its diagonal-line triangle and the points X and Y, the conics of the range with respect to which X and Y are conjugate points will coincide as also their poles with respect to XY, namely, Q_1 and Q_2.

Hence in that case all the conics of the range touch the locus of the poles of XY at the same point.

13. We may now generalise the theorems of § 9.

(*a*) Given a triangle and two points X and Y, the conics Σ of all the conics inscribed in the triangle cut harmotomically the conic through X and Y with respect to which the triangle is self-conjugate.

(*b*) Given a triangle and two points X and Y, the conics Σ of all the conics with respect to which the triangle is self-conjugate cut harmotomically the conic through X and Y which circumscribes the triangle.

(*c*) Given two points X and Y, and a conic S touching XY, the polar of the point of intersection of the other tangents from X and Y passes through the harmocentre of the triangle formed by any three tangents to S, and also through the pole of XY with respect to the conic through X and Y which circumscribes any triangle self-conjugate with respect to S.

(*d*) If X and Y are conjugate points with respect to a conic S, the pole of XY with respect to S lies on the conic through X and Y with respect to which any three tangents to S form a self-conjugate triangle, and also on the conic through X and Y which circumscribes any triangle self-conjugate with respect to S.

(*e*) If X and Y are two points, the harmocentre of the triangles formed by taking three out of four lines are collinear.

(*f*) The theorems correlative to the foregoing may be stated, but are not specially interesting.

14. If we project a pair of opposite vertices A, A' of the given quadrilateral to ω, ω', the range of conics becomes a system of confocal conics in association with X' and Y', the projections of X and Y.

The diagonal-line triangle of the quadrilateral projects into the line at infinity and the axes of symmetry of the confocals. Hence the σ conics of X and Y become parabolas touching the axes of symmetry, while the conic $P_1P_2AA'XY$ of § 3 becomes the circle on $X'Y'$ as diameter. Thus P_1' and P_2' are the points in which this circle cuts either the tangents from X' to the σ' of Y' or the tangents from Y' to the σ' of X'. Hence the angles at P_1' and P_2' are right angles, while Q_1' and Q_2' are the other two vertices of this quadrilateral of tangents, so that X' is the orthocentre of $Y'Q_1'Q_2'$ and Y' of $X'Q_1'Q_2'$.

Hence $Q_1'Q_2'$ is perpendicular to $X'Y'$, and, further, these lines divide harmonically the join of the foci S and H, common to the system. Also the parabolas touch $Q_1'Q_2'$. Further, from § 3, a conic can be drawn through $P_1'P_2'X'Y'SH$, and with respect to it SH and $X'Y'$ are conjugate chords.

15. The system of conics harmotomic to the Σ's of X' and Y' are all rectangular hyperbolas through X' and Y', having S and H as conjugate points; including the projection of the conic $XYUVW$, that is, the rectangular hyperbola through X', Y' and the middle point of SH and having its asymptotes parallel to the axes of symmetry. We may now restate the theorems of § 13 for this special case, but again the results are not specially interesting.

N. M. G.

MATHEMATICAL NOTES.

1124. ***Bernoulli's differential equation.***

Elementary students who have mastered the equation

$$y' + Py = Q \quad \text{.................................}(1)$$

often fail with the extension to

$$y' + Py = Qy^n, \quad \text{.................................}(2)$$

the preliminary transformation of the latter equation being to them only a trick which their memory cannot be trusted to retain. An alternative treatment of the extension will sometimes overcome this difficulty.

The solution of the original equation depends on replacing (1) by

$$(y' + Py)u = Qu, \quad \text{.................................}(3)$$

where u is so chosen that the left-hand side is the derivative of yu; that is, taking

$$u = e^{\int P\,dx},$$

we have (1) equivalent to

$$\frac{d(yu)}{dx} = Qu, \quad \text{.................................}(4)$$

which is integrable immediately. But by the use of the same factor, (2) becomes

$$\frac{d(yu)}{dx} = Quy^n,$$

that is,

$$\frac{d(yu)}{dx} = \frac{Q}{u^{n-1}}(yu)^n,$$

of which the integration is almost as obvious as that of (4).

E. H. N.

1125. ***Note on the Tetrad whose opposite joins are conjugate lines with regard to a given quadric.***

In Volume III of Baker's *Principles of Geometry* (p. 35) it is proved that if A, B, C, D be four points, such that two pairs of opposite joins of these be conjugate with regard to a given quadric, say CA conjugate to BD and AB conjugate to CD, then also the third pair, BC and AD, are conjugate lines with regard to this quadric.

In this note it is shown that of the four points, any three, say A, B, C may be taken arbitrarily and then the fourth lies upon a definite line. By considering the properties of this line a very concise and symmetrical proof of the theorem is obtained.

Let A, B, C be three arbitrary points and O the pole of the plane ABC with regard to the given quadric. Let the conic in which the plane ABC cuts the quadric be called S, and let the polars of A, B, C

with regard to S be the lines $B'C'$, $C'A'$, $A'B'$. The polar line of CA is OB'. Hence if BD is to be conjugate to CA, D must lie in the plane BOB'. By the same argument, if CD is to be conjugate to AB, D must lie in the plane COC'.

Hence if H be the point of intersection of the lines BB', CC' and we take D to be any point of OH, then A, B, C, D will be a tetrad of which CA, BD, and AB, CD are two pairs of conjugate lines with regard to the given quadric. The theorem that BC and AD are also conjugate lines follows immediately. For, by Hesse's theorem the lines AA', BB', CC' meet in a point and hence O, D, A, A' all lie in the same plane. This proves that AD meets OA', the polar line of BC, and therefore BC and AD are conjugate lines with regard to the given quadric.

R. J. LYONS.

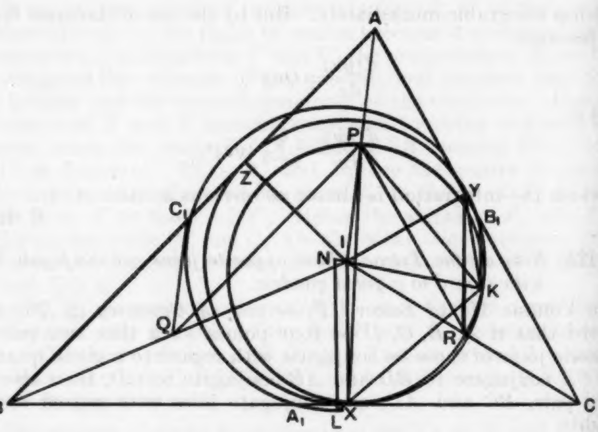
1126. *Two theorems on the geometry of the triangle, leading to a proof of Feuerbach's theorem.*

Let ABC be a triangle, $A_1B_1C_1$ its medial triangle, N its nine-point centre, I its incentre. Let the incircle touch the sides at X, Y, Z and let P, Q, R be the mid-points of IA, IB, IC.

(1) The circles on PX, QY, RZ as diameters, and the nine-point circles of BIC, CIA, AIB all meet the incircle at the same point K.

(2) This point K lies on the nine-point circle of ABC.

(3) The incircle and the nine-point circle of ABC touch at K.



(1) Let the circle on PX as diameter meet the incircle at K. Join PX, PR, PK, KY, RY, ZX, ZY. Then, since K is on the circle XYZ,

$$\begin{aligned}\angle XKY &= 180° - \angle XZY\\ &= 180° - (90° - \tfrac{1}{2}C)\\ &= 90° + \tfrac{1}{2}C,\end{aligned}$$

and $\angle XKP = 90°$; thus $\angle PKY = \frac{1}{2}C$. But because PR is parallel to AC, $\angle PRY = \angle RYC = \angle RCY$, because R is the mid-point of RC and $\angle IYC = 90°$.

Thus $\angle PKY = \angle PRY$, and so K lies on the circle PRY, the nine-point circle of CIA. Similarly K lies on the nine-point circle of AIB. In like manner the circle on QY as diameter and the nine-point circles of AIB, BIC meet the incircle in a point K_1, which is the same as K, since it lies on the incircle and the nine-point circle of AIB. Similarly K lies on the circle on RZ as diameter.

(2) Since K lies on the circle PYB_1R,

$$\angle PKB_1 = \angle PRB_1$$
$$= \tfrac{1}{2}A \text{ (since } PRB_1A \text{ is a parallelogram).}$$

By similar reasoning, $\angle PKC_1 = \frac{1}{2}A$.

Hence $$\angle B_1KC_1 = A = \angle B_1A_1C_1,$$

and so K lies on the circle $A_1B_1C_1$.

(3) Let KX meet the circle $A_1B_1C_1$ at L. Join NA_1, NL, A_1R, RX, A_1K.

$$\angle A_1NL = 2\angle A_1KL \text{ (angles at centre and circumference)}$$
$$= 2\angle A_1RX,$$

since K lies on the circle A_1QRX. But

$$\angle A_1RX = \angle RXC - \angle RA_1X = \tfrac{1}{2}C - \tfrac{1}{2}B.$$

Thus $$\angle A_1NL = C - B.$$

Hence NL is perpendicular to BC, for if AD is perpendicular to BC then $\angle A_1ND = 2C - 2B$. Thus NL is parallel to IX, and so NIK is a straight line. That is, the circles XYZ, $A_1B_1C_1$ touch at K.

The following theorems in connection with the figure are in some cases interesting and can be proved by simple geometry:

(1) PX, QY, RZ are concurrent. The coordinates of the point of concurrence are given by

$$\frac{a\alpha(s-a)}{(b+c)(2a+b+c)} = \text{two similar expressions.}$$

(2) If N_1, N_2, N_3, are the nine-point centres of the triangles BIC, CIA, AIB, the triangle $N_1N_2N_3$ is similar to the triangle XYZ.

(3) I is the orthocentre of the triangle $N_1N_2N_3$.

(4) N is the circumcentre of $N_1N_2N_3$.

(5) The circumradius of $N_1N_2N_3$ is $\frac{1}{2}OI$.

(6) If parallels to N_2N_3, etc., are drawn through N_1, etc., to form a triangle $M_1M_2M_3$, the circumcentre of ABC lies on the circle $M_1M_2M_3$.

(7) The triangles AIN_1, OIX are equal in area.

(8) The circle KIN_1 touches NN_1 at N_1.

(9) If IN_1, IN_2, IN_3 meet BC, CA, AB at X_1, Y_1, Z_1 respectively, X_1, Y_1, Z_1 lie on a straight line, which is the common tangent to the nine-point and inscribed circles of ABC, and

$$BX_1 : X_1C = a - b : a - c.$$

W. F. Beard.

1127. ***Conormal points on an ellipse.***

1. Burnside's formula for the concurrence of normals at the points on the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$
whose eccentric angles are α, β, γ, in the form
$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$$
is generally proved by a rather clumsy trigonometrical analysis, which deserves to be replaced by something better. The following proof is fairly elegant and is based on a well-known property of the rectangular hyperbola, namely, that the orthocentre of any inscribed triangle lies on the same hyperbola.

If P, Q, R, S be a set of conormal points with eccentric angles $\alpha, \beta, \gamma, \delta$ such that the normals at these points pass through (x_1, y_1), then P', Q', R', S', the points on the major auxiliary circle corresponding to P, Q, R, S, lie evidently on the rectangular hyperbola
$$a^2x_1y-aby_1x=(a^2-b^2)xy. \quad \text{......(i)}$$
The orthocentre of $P'Q'R'$ is
$$a(\cos\alpha+\cos\beta+\cos\gamma), \quad a(\sin\alpha+\sin\beta+\sin\gamma)$$
and lies on (i), by the property mentioned above. Hence
$$ax_1(\sin\alpha+\sin\beta+\sin\gamma)-by_1(\cos\alpha+\cos\beta+\cos\gamma)$$
$$=(a^2-b^2)(\cos\alpha+\cos\beta+\cos\gamma)(\sin\alpha+\sin\beta+\sin\gamma). \quad \text{......(ii)}$$
Again, substituting for (x, y) in (i) the values $(a\cos\alpha, a\sin\alpha)$, $(a\cos\beta, a\sin\beta)$, $(a\cos\gamma, a\sin\gamma)$ in succession and adding, we get
$$ax_1(\sin\alpha+\sin\beta+\sin\gamma)-by_1(\cos\alpha+\cos\beta+\cos\gamma)$$
$$=(a^2-b^2)(\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma). \quad \text{.........(iii)}$$
From (ii) and (iii) we get at once
$$(\cos\alpha+\cos\beta+\cos\gamma)(\sin\alpha+\sin\beta+\sin\gamma)$$
$$=\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma$$
or
$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0. \quad \text{...............(iv)}$$

2. If we replace (i) by the more general form
$$A(x^2-y^2)+2Hxy+2Gx+2Fy+C=0$$
and proceed as before, the modified result is
$$A\{\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)\}$$
$$+H\{\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)\}=C/a^2. \quad \text{......(v)}$$
In particular, if α, β, γ be the eccentric angles of three points on an ellipse of semi-axes a, b, whose θ-normals are concurrent,
$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=2ab\cot\theta/(b^2-a^2). \quad \text{...(vi)}$$
(*Vide* Casey's *Analytical Geometry*, 2nd edition, 1893, p. 539.)

A. A. Krishnaswami Ayyangar.

REVIEWS.

Foundations of Point Set Theory. By R. L. Moore. Pp. v, 486. \$5.00. 1932. American Mathematical Society Colloquium Publications, 13. (American Mathematical Society)

This book is clearly not intended for the novice, nor is it a text-book on Point Sets. It is rather the bringing together into a logical whole of a vast collection of researches carried on by the author and his pupils. A commendable effort is made to reach the utmost generality and an absolute minimum is assumed about the " points " about which the argument revolves. Axiom 1 concerns collections of regions (not defined except that they are point sets—axiom 0) and is a very general form of Borel's theorem for sequences of closed sets. This alone carries one through 86 pages of the most closely packed reasoning, concerning continua and connections. Axiom 2 is that if P is a point of a region R there exists a non-degenerate connected domain containing P and lying wholly in R. This begins to restrict the kind of " space " in which the " points " lie, and as far as page 151 the theorems mostly deal with continuous and branched arcs. The interior of a Jordan simple closed curve arrives on page 163, after two more axioms have been allowed entry. The remainder of the book can be judged to some extent by theorem 8 on page 417 : If the bounded simple closed curve g has only a finite number of points in common with the bounded simple closed curve $ABCDA$ and does not contain A, B, C or D, then the interior of $ABCDA$ can be divided by a double ruling . . . into subdivisions, such that the interior of each of them is either wholly within or without g.

On page 430 the idea of distance is mentioned, and it is shown what the previous theory becomes, when it is applied to so concrete a geometry.

The author has set out to achieve something very valuable and has succeeded, but one cannot help wishing that he had made the book more readily available to the ordinary mortal. In the first twenty pages or so, and often throughout the book, elementary examples illustrating what is being proved would have been helpful. Indeed an introductory résumé of the results as applied to ordinary points, with statements of those properties which are actually irrelevant, would have enabled the student to appreciate the advantage of the extreme degree of abstraction used.

On page 10 a misleading misprint occurs. In line 4 of the proof of theorem 13 the last α should be γ and in line 7 α should also read γ. On page 12 in the top few lines an indication of proof required to justify the statement should have been given.

In axiom 1 the term " sub-collection " is not defined, and this is very unfortunate as it appears to mean a collection of regions each of which is a subset of some regions of G_n. For example a sub-collection of the regions $(x-1/n)^2+y^2<1$ would be the collection of the regions

$$\max(|x-1/n|, |y|)<\tfrac{1}{2}.$$

This is an uncommon use of the term sub-collection and yet the whole book is based on axiom 1 and therefore on this term.

A book to be recommended only to the specialist in generalized topology.

P. J. D.

Introduction to General Topology. By W. Sierpiński. Translated by C. Cecilia Krieger. Pp. x, 238. 17s. 1934. (University of Toronto Press: Oxford University Press)

This book is the second volume of a work by Professor Sierpiński on the Theory of Aggregates. It is, however, complete in itself to one who is already familiar with the use of the infinite cardinals and ordinals and the methods of transfinite induction. For the benefit of the reader who lacks such knowledge,

an Appendix by the translator provides a brief introduction to the theory which is required in the text.

Professor Sierpiński's book gives an axiomatic development of the elements of the Theory of Topology, with a general discussion of sets and spaces. Only on the very last page is the idea of a curve introduced, and then merely as a definition and with no applications. The open set is taken as the fundamental concept, and the author justifies his choice of this development as being "simpler and more intuitive than other axiomatic treatments". The theory is developed from an account of the simplest properties of open and closed sets, by means of the successive introduction of the Axioms of Open Sets, Countability and Regularity, into a detailed treatment of the properties of Metrical Spaces and Analytical Sets. The axioms are introduced as they are needed for new definitions and theorems, so that, in general, the scope of the discussion narrows as the complexity increases. The method is a powerful one, attractive by reason of its logical simplicity, and in Sierpiński's hands it has the effect of making the whole development seem completely natural and indeed inevitable.

It is a matter for regret that there is no bibliography. The footnotes are hardly sufficient for the student who wishes to trace the frequent references to original work which occur throughout the text.

The Appendix makes no claim to be anything but a very brief outline of the theory of transfinite numbers. One cannot imagine that it would satisfy a reader, even a novice, who is prepared to follow the close logical arguments of the main text, passing over, as it of necessity must, all the difficulties of the Multiplicative Axiom and the Theory of Types. Here again, a bibliography would have been helpful.

The translation betrays itself occasionally by erratic punctuation and odd phraseology, and one notices a number of typographical errors. The Table of Contents is full, but the Index is inadequate. M. E. G.

Theorie der Funktionen mehrerer komplexer Veränderlichen. By H. Behnke and P. Thullen. Pp. vii, 115. RM. 13.80. 1934. Ergebnisse der Mathematik, Band III, Heft 3. (Springer)

This excellent report of a hundred odd pages on the main topic of the theory of functions of several complex variables is a kind of complement to Osgood's classical *Lehrbuch der Funktionentheorie, II*, giving a concise but readable résumé of the results obtained in recent years. In view of the fact that in the recent literature both point of departure and method vary considerably with the authors, every researcher in this field will appreciate the effort that produced this tract. The method of the authors is direct, that is, without reference to real or imaginary parts. As a rule proofs are only sketched but full references are given.

In the last four or five years the number of publications on the subject has increased beyond proportion, the rather facile generalisations have given way to front attacks on the specific difficulties encountered especially in the description of domains of regularity and singularity. The notion of domain itself had to be adjusted to the new conditions. This applies in particular to the idea of a Riemann surface over the "space" of n complex variables. This is given in the first chapter, together with a meticulous development of the new notions, as, the infinite element, domains of regularity, meromorphy, normality, analytic (pseudo-conformal) mapping, etc. The second chapter introduces the reader to the geometrical language, *i.e.* to analytic surfaces, proper and improper boundary points, essential and non-essential boundary points, analytic Gebilde, analytic hypersurfaces, and for two independent variables, to the notions of some special regions like bicylinder (Dizylinder), Reinhardt's Körper, Kreis-, Hartog's- and Cartan's Körper.

After this introductory part the next three chapters deal with the representation of regular functions by Taylor series and by Cauchy's integrals (written

out in two independent variables), with singular manifolds and analytic continuation, natural boundaries included, with distribution of zeros and non-essential (polar) singularities, with functions possessing prescribed zeros and poles (extensions of Mittag-Leffler's theorem and of Weierstrass' factorisation theorem).

In chapter 6 we get a clear presentation of the important notion called Regularitätshülle, the minimum domain of regularity containing a given domain, together with the theory of domains of convergence and of normality leading to the extension of Runge's theorem to several variables. The last chapter deals with the recent results of H. Cartan, Carathéodory, Bergmann and others on the mapping of the new complicated domains on simpler domains. This chapter is most valuable on account of the thoroughgoing discussion of the intricate arguments involved. P. D.

Relativity Thermodynamics and Cosmology. By R. C. Tolman. Pp. xv, 502. 30s. 1934. International series of monographs on physics. (Oxford)

About five years ago, general relativity, apart from the more general unified field-theories of Einstein and others, was to all appearance a worked-out subject. A remarkable revival came when mathematical physicists began to realise the significance of the work of Friedmann, Lemaître, Tolman and others, on non-static models of the universe. Out of their work grew rapidly the relativity theory of the expanding universe with its far-reaching implications in cosmical physics. Consequently there is now a demand for a book which will give an up-to-date account of those parts of general relativity which have proved most important in these studies, and which will serve as a guide through the extensive literature of the subject, providing at the same time an epitome of its results.

Professor Tolman's book will go a long way towards meeting this demand. Its title, to be sure, presents a puzzle reminiscent of a school examination question, " Punctuate the following sentence . . . " ! One soon discovers, of course, that the book is not intended to contain separate treatises on relativity, thermodynamics, and cosmology, but rather to be a study of those parts of any one of these subjects which bear upon any of the others, the goal being the union of all three.

The main divisions of the book are one on special relativity and its relations to mechanics, electromagnetism, and thermodynamics, one on general relativity and its relations to these topics, and finally one devoted to applications of the previous work to cosmology. All this occupies some five hundred pages, and one's first impression is that it could have been cut down to half that length by judicious compression of argument and quotation from existing standard authorities. On second thoughts, however, one decides that the author has catered for a reader who has but little previous knowledge of relativity, but who yet wishes to gain a fairly detailed knowledge of relativistic cosmology.

This type of reader must surely be a rarity, for anyone who is not to some extent a specialist in relativity would probably desire no more than a brief description in general terms of the methods and results of relativistic cosmology. On the other hand, anyone wishing to learn relativity from the start would scarcely be well-advised to begin with this book, for, though it naturally has to discuss for example the Lorentz transformation and Einstein's field equations, it does not give actual derivations of either.

Nevertheless Professor Tolman has given us an excellent book. Any relativist who reads through its discussion of many results, which in themselves are familiar to him, will benefit much by what is one of the author's important contributions, namely an insistence on the physical significance of each step taken, and a devising of methods which put this significance in the foreground.

Another of Professor Tolman's contributions, probably his chief one, which receives a full exposition in this book, is his formulation of the second law of

thermodynamics in general relativity. He demonstrates clearly the necessity of some such form for cosmological applications, since an application of the unmodified classical law to the universe as a whole can lead to erroneous conclusions regarding the reversibility of cosmical processes. Indeed Eddington stressed in his Gifford Lectures (1927) the need which then existed in this connection for a general relativity system of thermodynamics. But in spite of Professor Tolman's important work towards remedying this need much yet remains to be done. His work still leaves open the question as to why an observer in the universe affixes " time's arrow " in one direction rather than in the opposite direction.

As regards relativistic cosmology apart from thermodynamic considerations Professor Tolman has given a comprehensive survey of the present state of the theory, and a very useful indication of the most profitable lines of future observational research suggested by it. This part of the work will no doubt remain for some time a standard of reference. It does nevertheless leave itself open to one criticism. A treatment of such very recent work developed by a large number of contemporary investigators might be expected to give a more explicit appreciation of their various contributions, together with a fuller bibliography, than has actually been supplied. To give a couple of examples, Heckmann's pioneering work on the sign of the curvature of space and the sign of the cosmical constant in an expanding universe, and the pioneering work of Whittaker and his school on " spatial distance ", receive no mention in the sections devoted to these questions. Or, again, the important cosmological work of Milne deserves more notice than a casual footnote.

It must be said in conclusion that neither author nor printers have spared themselves trouble in giving us a beautifully produced book. In spite of any criticisms here offered, one hopes that the book will be given the important place which it deserves in all libraries of mathematical physics.

W. H. McCrea.

The Theory of Electric and Magnetic Susceptibilities. By J. H. Van Vleck. Pp. xii, 384. 30s. 1932. International series of monographs on physics. (Oxford; at the Clarendon Press)

The great revolution in physical theory now seems to be over and we have reached a breathing space which provides an opportunity for a general stock-taking. A few years ago many books were rushed through the press in the fear that they might be out of date before publication, but there are parts of the quantum theory which have now reached a certain stability, and books concerning them may be purchased with the confidence that they will remain authoritative for a long time to come.

This is true of the volume before us. It deals with the theory of dielectric constants and of magnetic susceptibilities and so concerns the properties of the *outside* of atoms and molecules. While the theory of the nucleus may yet undergo many violent changes, the quantum theory of the electrons surrounding it seems to be capable of answering all the questions asked of it, and for that reason is not likely to suffer any radical modification.

This book is an authoritative account of an important field of study. It begins with a thorough examination of the classical foundations of the theory of electric and magnetic susceptibilities, and though this may scare the non-mathematical reader, it forms the solid foundation on which the book is built. Then there is an account of the classical theory as developed by Langevin and Debye and the method by which dipole moments are determined. Then follows a characteristic introduction to quantum mechanics (Chapter VI), which includes a demonstration of the equivalence of the definitions and concepts of wave mechanics and matrix mechanics. The object of this chapter is to develop the machinery for calculating energy levels, such as are required in the study of electric and magnetic susceptibilities, and this involves the de-

velopment of perturbation theory and the theorem of spectroscopic stability. The result is a somewhat formal account of quantum mechanics, but as the author in extenuation thereof says, "in the last analysis a theory is most 'physical' when it permits the calculation of a large number of experimentally observable quantities in terms of a few fundamental postulates".

The susceptibilities obtained by the new quantum mechanics are more closely akin to those of the classical theory than those obtained by the old quantum theory. Thus all theories predicted that molecules, possessing a permanent electric dipole μ, would have an electric susceptibility given by a formula of the form

$$\chi = N(a + c\mu^2/kT),$$

in which c is a pure number. The classical theory (1912) gave $c=\frac{1}{3}$. The first calculation by the orbital quantum theory, using integral quantum numbers (1921) gave $c=1{\cdot}54$, while the second, using half integral quantum numbers (1925), gave $c=4{\cdot}57$. The new mechanics in 1926 finally restored c to its original value of $\frac{1}{3}$.

In the later parts of the book a theory is developed of the magnetic moments of the iron group of atoms and ions and an explanation is provided of the long established fact that the effect of the orbital angular momentum is quenched. In the development of this subject the author himself has made important contributions, and the account is consequently clear and exact. The book includes a report of the work of Heisenberg and others on the theory of ferromagnetism, which has at last elucidated the mystery of the Weiss molecular field.

Altogether this volume is a scholarly and masterly piece of work, which is likely to establish it as a standard book of reference. It requires close study, but the reader may be confident that it will be well repaid.

A Mathematical Treatise on Vibrations in Railway Bridges. By C. E. Inglis. Pp. xxv, 203, 21s. 1934. (Cambridge)

This mathematical treatise is a brave attempt to solve the difficult problem of the vibrations in railway bridges, caused by the passage of locomotives over them, by the use of mathematics modified by common sense, or, shall we say, by engineers' mathematics. The finding of the vibrations of a bridge, regarded as a uniform girder, when it is traversed by a locomotive which has part of its mass mounted on springs and which applies a hammer blow to the bridge as it moves, that is essentially the problem attacked in this book. This problem, if treated with mathematical rigour, leads to all sorts of difficulties, some of which are purely mathematical and some due to the great number of data "which insist on being taken into account", as the author says in his Introductory Chapter. In this same chapter we read:

"The first really comprehensive attempt to put the problem of bridge impact on a scientific basis was made by the Bridge Stress Committee, which was established under the chairmanship of Sir Alfred Ewing in March, 1923, and published its report in October, 1928. Previously, although a great mass of experimental data had been accumulated, in default of any underlying theory ... the inferences which could be drawn from this welter of disconnected experimental results were disappointingly meagre.... Mathematical analysis is required to indicate the lines along which experiments should proceed, and experiment, in its turn, is necessary to check the validity of theoretical predictions and to prevent mathematics from running off the scent and barking, so to speak, up the wrong tree."

"As a member of the Bridge Stress Committee, the writer was engaged in evolving a satisfactory system of mathematical analysis, moulding it into shape till it fitted the facts, and pruning away unnecessary excrescences until it became a tool of reasonable simplicity and practical utility. Without the guidance of experimental results to point the way at every cross-road and to

check the validity of analytical predictions, this task would have been quite impossible and, deprived of the experimental signposts, any mathematician, no matter how astute he might be, would inevitably lose his way in the labyrinth of side tracks which confront the traveller in this comparatively unexplored territory ".

Thus we see what the author set out to do and the way in which he went about it. He has done his task very skilfully ; but, of course, after his own confession we can hardly accept the good agreement between his analytical results and the few experimental observations given in the book as a convincing justification of all his moulding and pruning of the rigorous mathematics. This is not to say that we think it is not justifiable. We think it is ; but we think the justification is in the mathematics itself. In fact, Professor Inglis has himself justified his pruning by mathematical arguments with, necessarily, constant references to realities to find what quantities were, or were not, negligible. The evidence in the book is such, indeed, that we cannot believe that the author could not have arrived at the same results if he had had no experimental bridge vibrations before his eyes, although probably he would have taken a longer time about it.

Let us repeat that the work is skilfully done. One way of attacking the problem would have been to write down at once the complete differential equations for the vibrations of the bridge and locomotive and then try to solve them. This, however, would have dismayed any student, as well as most mature investigators. Professor Inglis's method has been to take the problem to pieces and add the difficulties to the problem one at a time. He starts by finding the deflections of beams under static loads and shows how to use harmonic analysis to find these deflections ; and this method of harmonic analysis is a powerful implement which is used throughout the book. In successive chapters the problem of a real locomotive on a real bridge is gradually approached by the taking into consideration in turn, of a moving load of constant magnitude, of an oscillating load, of a combination of these two loads, of a damping resistance along with the loads and, finally, of the vibrations of the spring-borne part of the locomotive itself. Thus, by easy stages, the reader is led to the problem in all its fullness, and at each step everything is discarded that does not affect the result to any appreciable degree.

That the complete problem leads to a formidable amount of arithmetic can be seen by a glance into Chapter IX, where only the results of the arithmetic are shown and nothing is visible to indicate the tremendous amount of fagging necessary to get these results.

There is one point in the book which a mathematician cannot fail to criticise, and that is the want of precision in the use of the word *resonance*. It is assumed that resonance (which presumably means the most violent oscillations) occurs when the period of the moving hammer blow, expressed by $P \cos 2\pi Nt$, is the same as that of the fundamental mode of oscillation of the bridge, which is expressed by $c \cos 2\pi n_0 t$. This gives $N=n_0$. But, by the very harmonic analysis used in this book, the term $P \cos 2\pi Nt$ gives rise to two terms with factors $\cos 2\pi(N-n)t$ and $\cos 2\pi(N+n)t$ in the applied forces, where n is proportional to the speed of the locomotive. If the harmonic method is thoroughly valid it must follow that such resonance as there is occurs when $N-n=n_0$. In the practical example given in the book, where damping is taken into account also, the assumption probably makes very little error because n is small in comparison with N. But it is conceivable that cases might arise in which the error would be significant.

One further point. In many places in the book seven-figure numbers are used in calculations arising from one- and two-figure data. The final results generally come back to two figures, which are precise enough to satisfy any engineer. Much labour could be saved by the retention of not more than four figures in the intermediate steps of the calculations. JOHN PRESCOTT.

Graphische Kinematik und Kinetostatik. By K. Federhofer. Pp. vi, 112. RM. 13.15. 1932. Ergebnisse der Mathematik, Band I, Heft 2. (Springer)

This is an extremely interesting book on a subject which falls rather outside the range of mathematics normally studied by engineers. It considers first of all methods for describing graphically the movements and accelerations of points, rigid laminae and plane linkages. The three-dimensional cases are then taken up in their turn. Some of these methods are merely convenient tools but others are related, as one might expect, to geometrical theorems of general interest, more particularly where the degrees of freedom of linkages are dealt with. Then on this foundation it is possible to make constructions for acceleration-reactions in a system, using also the well-known methods of graphical statics. But when it comes to three-dimensional problems of this kind the reviewer feels, though it may be prejudice, that the tools are too simple for the complexities present. If not, there remains a field here which has, as yet, been explored only partially and the geometer may agree that after all there may be something in this applied mathematics. In any case the book should be in the libraries of all engineering departments. P. J. D.

Elementary Dynamics. By R. C. Gray. Pp. xi, 211. 5s. 1934. (Macmillan)

The preface states that in this book dynamics is treated as a branch of applied science and not as a branch of mathematics. We suppose that the author means by this that he has chosen applications of dynamical principles from practical engineering problems. Certainly he has succeeded in imparting a practical bias throughout and he has chosen the examples very well indeed. Taken as a whole we found the book interesting and pleasingly set out. It starts from Matriculation standard and finishes well beyond Intermediate standard, the last few chapters being on the motion of a rigid body. There are chapters on the equilibrium of forces and on centroids which more usually occur in books on statics.

We do feel, however, that there are too many statements without proof, and definitions and proofs which are inadequate. Thus on p. 7, it is stated, without proof, " the area enclosed by the speed-time graph and the time axis is a measure of the distance run ", and on p. 8 velocity is defined by " the speed, and the direction of motion of a particle, are called its velocity ".

On p. 18 it is stated that, since acceleration is a vector, two or more accelerations can be compounded into a resultant acceleration ; this argument one sees far too often in modern books, and its insufficiency is easily seen by considering rotations ; although a rotation of any magnitude about an axis can be represented by a directed line, only infinitely small rotations can be compounded vectorially.

Angular velocity is defined for a point instead of for a line, and although radians are not mentioned in the definition they are used in obtaining the formula $v=r\omega$. The acceleration of the bob of a pendulum is stated to be $g \sin \theta$ and its motion is discussed before forces have been mentioned.

Throughout the book symbols have been supposed to include units and so the so-called Stroud system is used, but no indication is given in the chapter on forces how to convert lb.-ft./(sec.)^2 to lb.-wt. or poundals. In many of the examples on forces, they are not resolved in the directions which give the simplest equations.

The direct use of calculus is avoided but the dot notation for rates and the fundamental ideas of the calculus are used throughout, as is inevitable in any book on dynamics. There are some most ingenious examples of calculus dodging in the chapters on centroids and moments of inertia. Calculus notation is used in finding the force on a table when a vertical chain falls on it, and in finding the tension in a belt over a rough pulley. In both cases dt, or dT, is

taken as a quantity whose square can be neglected, and in finding the tension of the belt cos θ is taken equal to 1, when the whole point is that the difference between cos θ and 1 is of the order of θ^2.

In the chapter on motion of a rigid body there is a bad mistake. It is said that rotation of a rigid body round *any* axis A can be obtained from

$$L_A = I_A \dot{\omega}.$$

The axis must, of course, be through the centroid or the instantaneous centre.

We have given sufficient instances to show that this book, although very interesting from the practical point of view, is not a model of accurate statement on the theoretical side. It is very well produced and we have found only one trivial misprint.
H. V. Lowry.

Differential Equations. By H. B. Phillips. Third edition. Pp. vi, 125. 10s. 6d. 1934. (Chapman and Hall)

The first edition was reviewed in the *Gazette* for December, 1922, and the second in March, 1926. The third edition has nine pages more than the second, with larger pages. As before, the book contains four chapters, of which the last deals with linear equations with constant coefficients, but the number of sections has risen from 42 to 49. The price has been increased from 6s. 6d. to 10s. 6d. The collection of problems, already good in the previous editions, has been again extended.
H. T. H. P.

Méthodologie Scientifique—Méthodologie Dynamique interne. (Archives de Philosophie Vol. X Cahier III). By J. de la Vaissière, S.J. Pp. 109. 24 fr. 1933. (Beauchesne)

To give an adequate treatment of scientific method, an author should possess some acquaintance with all the sciences in addition to a thorough training in logic and philosophy. It is difficult for any one man to be so completely equipped, and indeed some books on the subject appear to have been written by those who have no first-hand knowledge of any branch of science. Father de la Vaissière, to judge from internal evidence, has a good knowledge of mathematics, of the older kind of mathematical physics, and of some aspects of psychology. He relies very much upon authority, and the best parts of the book consist of quotations from Poincaré, Claude Bernard, and other writers, nearly all French and not very recent. The account of modern developments is inadequate. For example, the principle of determinism is stressed, but except by following up a reference to Eddington in a footnote, the reader would have little chance of learning that this principle had ever been challenged. The description of experimental psychology as a purely qualitative science seems to show a total ignorance of recent English and American investigations.

However, in spite of these defects, the book gives an interesting discussion of the steps logically necessary in the establishment of a science, and it also shows the chain of thought actually followed by scientists in reaching their conclusions. Formal logic and the Baconian rules of induction, though useful in testing results once they have been obtained, are of little value in obtaining new results. In the process of discovery the subconscious mind often plays a large part. A mathematician who has unsuccessfully studied a problem one night, may have the solution suddenly flashed into his consciousness while shaving next morning.

Teachers will be specially interested in Chapter IX (*Formation de l'esprit scientifique*), which includes an account of an enquiry made between 1896 and 1900 into the French system of teaching. It was found that the highest places in the mathematical examinations of the École Polytechnique were taken by those who had received the best training in literary subjects. The explanation of the apparent paradox seems to have been that the examinations were largely a test of memory, in which the greatest success was obtained by those who learnt their books and lecture notes by heart and reproduced them

exactly, without attempting to think for themselves. It would be interesting to know how far conditions have changed since 1900, but at any rate, when we are told of the evils of the unreformed Cambridge Mathematical Tripos, which neglected important aspects of the subject and placed great emphasis on the solution of somewhat artificial problems, we may remember that there were defects at least as serious, if of a different kind, in other examination systems.

H. T. H. P.

Elementary Calculus. II. By C. V. Durell and A. Robson. Pp. xii, 272. 7s. 6d. Without appendix of further examples, 6s. 6d. 1934. (Bell)

This book is not quite so successful as its predecessor, Vol. I. The latter was intended only as an introduction and could dispense with the more rigorous type of proof, concentrating rather on making the subject reasonable to the reader. The second volume deals with more advanced topics and the lack of rigour is more serious. The authors have also felt it to be necessary to include all the topics which might be considered at this stage. The result is too condensed and is more suitable for the examinee than for the intelligent student. For example, envelopes, exact differential equations and the general linear differential equation might well have been omitted. Evolutes can be handled without envelopes. At the most, envelopes of families of straight lines would have been ample. The book also deals somewhat sketchily with such topics as double integrals and partial differentiation. Unfortunately many examiners are still bound by a tradition of great wealth of technical ability with little logical basis and the authors do their best for the candidate.

The book starts at Chapter XII (Chapter XI is logically the end of Vol. I) with the integral for the logarithm but loses its courage and *proves* the coincidence of this integral with the usual index-logarithm. It nowhere explains to the student what is meant by a power with an irrational index. It takes this for granted. On the other hand the authors are to be praised for putting hyperbolic functions in their logical place in Chapter XIII instead of following the usual and bad practice of relegating them to an appendix.

In the chapter on Partial Differentiation it surely would have been better to confine attention to functions having a total differential, that is those for which

$$\delta z = A\,\delta x + B\,\delta y + \eta(|\,\delta x\,| + |\,\delta y\,|),$$

where η tends to zero with $|\,\delta x\,| + |\,\delta y\,|$, and A, B are independent of δx, δy. This avoids the cumbrous use of the mean-value theorem in a proof which is incomprehensible to the type of student for whom the rest of the book is suitable.

In the chapter on differential equations too much attention is given to the general linear equation and too little to the equation with constant coefficients which is far more important at this stage.

On the whole this volume may be suitable for rapid reading but it should be supplemented by books which will correct the all too common idea that anything expressed in " mathematical " symbols must have a meaning and be true.

P. J. D.

Higher Certificate Calculus. By C. V. Durell and A. Robson. Pp. xii, 241-368, xvi. 4s. 1934. (Bell)

This is an extract from the author's *Elementary Calculus*, Vol. II ; and " is designed to meet the extra requirements of the easier papers in higher certificate examinations ". It contains work on integration, the logarithmic, exponential and hyperbolic functions, geometrical applications of the integral, and differential equations. The first part, Chapter XI (Integration by substitution and by parts) and Chapter XII (Logarithmic and exponential functions) can be obtained separately, price 1s. 6d.

T. A. A. B.

Das bizentrische Viereck. By F. Bücking. Pp. iv, 44 and 6 plates. RM. 3.60. 1933. (Teubner)

Professor Bücking's monograph is a contribution to the theory of Poncelet's polygons. The question which Poncelet answered is this: can a polygon of an assigned number of sides be inscribed in one given conic and circumscribed to another? He showed that unless the two conics are related in a particular way there is no solution and that, if the conics are so related, then *any* point on the circumscribing conic may be taken as an angular point of the polygon.

The author begins by finding the condition necessary for a quadrilateral to be inscribed in one given circle and to circumscribe another. Such a quadrilateral he calls *bizentrische*. In the early stages he uses simple trigonometry, and, as a result, the way is long and wearisome. The large number of misprints makes it all the more tiring. Even the author's *hauptgleichung* is misprinted (p. 6), an error which should have been obvious to anyone reading the proofs since, in Euclidean geometry, all equalities involving length are homogeneous in length. The *hauptgleichung* $OM = \sqrt{\{r^2 + \rho^2 \mp \rho\sqrt{(4r^2 + \rho^2)}\}}$ gives the distance between the centres of the circles. Three inversions are then made use of and we are introduced to the six plates of well-drawn but complicated diagrams at the back. These diagrams open outwards to the right and may be scanned with ease while the text is being read—by this device the need of constantly turning back is dispensed with. The author then passes on to deal with the case of conics, but leaves us with the feeling that the work could have been more concisely set out with the aid of elliptic functions.

Those who are specially interested in Poncelet's polygons will find the monograph very useful. Many writers are quoted but the author does not make it clear to what extent he has relied on their work. V. N.

The Essentials of School Geometry. By A. B. Mayne. Pp. xiv, 409, ix. 4s. 6d. 1933. (Macmillan)

The following is extracted from the preface: "Those teachers and examiners who are in a position to compare the results obtained by the teaching of Geometry in schools to-day, with those obtained before the dethronement of Euclid, agree almost unanimously that there have been both gain and loss. On the one hand, almost all pupils to-day acquire much more power in applying and reasoning from the fundamental facts of Geometry than did their predecessors, but, on the other hand, their reasoning is often less rigorous and the average pupil often fails lamentably to reproduce the standard proofs when called upon in examinations. Almost all would agree that the gain outweighs the loss; for the educational value of the subject lies far more in the former than in the latter accomplishment. There are many, however, who think that the loss need not accompany the gain, and one of the objects of this text-book is to help the average pupil by providing proofs of the standard theorems which, within limits, shall be accurate and rigorous."

This book lives up to the wise words of its introduction. Unlike some modern Geometry books, the greatest care is taken over the style of setting out the standard theorems. The author is especially careful with proofs depending on congruent triangles, and with regard to references to the congruency theorems (thus he insists on "two angles and *corresponding* side," not the common slipshod "two angles and one side"). Presupposing a preliminary experimental course, it is an admirable book up to the stage of the School Certificate and just beyond. The logical foundations are carefully stated at the outset (axioms, postulates and all), the development is clear, the writing attractive, and the examples numerous, interesting, and well-graded. No boy or girl trained on the lines of this book should have any difficulty in recognising a geometrical proof when he sees one, or should fail to know the meaning of "arguing in a circle", or "begging the question". It is almost invidious to

particularise good points. Special attention may, perhaps, be called to the interesting and careful section on loci connected with the circle, pp. 264 to 268.

One proof is open to criticism, viz. that of the theorem, "If the straight line joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic."

There are three methods of proof to be found in text-books:

(i) If the circle through ABC does not pass through D, let it cut AD or AD produced in E, etc.

This is common, but slipshod. See (iii) below.

(ii) It is not possible for BC to coincide with BD, for, if so, etc. Suppose that angle ABC is greater than angle ABD. Since BD lies in the angle ABC, the circle ABC (if it does not pass through D) must cut BD or BD produced. Hence, etc.

This is Mr. Mayne's method, and it is to be found in other books. It is better than (i). Its weakness is that it makes use of an unformulated property of the circle.

(iii) If circle ABC does not pass through D, it must cut AD between A and D, or in AD produced, or in DA produced, or AD must be a tangent to the circle. Each of these cases can be examined and shown to be impossible. This is the only really satisfactory method. The trouble is that it cannot be taken till after the tangency proofs: hence the theorem becomes separated from that to which it is converse. But this seems a lesser evil than to base the proof on a property of the circle which may be more or less vaguely intuitive but is not formally stated. H. E. P.

(1) **Analytic Geometry.** By F. S. Nowlan. Second edition. Pp. xii, 352. 13s. 6d. 1934. (McGraw-Hill)

(2) **Analytical Geometry.** By V. C. Poor. Pp. v, 244. 13s. 6d. 1934. (John Wiley and Sons; Chapman and Hall)

These two volumes, which cover the same ground, are designed for junior students in American and Canadian universities. There are short early chapters on Loci and Polar Coordinates and later chapters on a few higher plane curves; and there is also some three-dimensional geometry. As in many English text-books there is insufficient regard for symmetry and notation. Both authors hesitate to assume familiarity with differential calculus although that subject surely comes at least a year before the general conic in a mathematical education.

Dr. Nowlan's book has a number of good features; it is carefully written: for example in the locus chapter the difference between

$$x+y=xy \quad \text{and} \quad 1/x+1/y=1$$

is mentioned and some consideration is given to such equations as

$$f(x, y)/g(x, y)=0 \quad \text{and} \quad \{f(x, y)\}^{2n}=0.$$

The conic is defined by the focus-directrix property and the development therefrom is quite logical; it is rightly pointed out that, with this definition, a circle is not a conic.

Dr. Poor's book is less accurate: the polar r is allowed to be negative, but a few pages later we find $r=\sqrt{(x^2+y^2)}$, and with the focus-directrix definition we are told that "the necessary and sufficient condition that an equation represent a conic section is that the equation be of the second degree". One resents being told to "find the data" (p. 107), and the use of

$$Ax^2+Bxy+Cy^2+Dx+Ey+F$$

instead of $ax^2+2hxy+by^2+2gx+2fy+c$ is very tiresome. On the other hand there are more examples in this book than in Dr. Nowlan's. The problems, we are told in the preface, have been so selected that some general principle must be used in their solution. Actually it is difficult to discover anything worthy of the name of "problem", unless it be to find the meaning of this example

(p. 152, 13): "What conditions on the general conic are necessary, so that secants through three of its points will determine it?"

The following construction for a hyperbola is worth quoting: "To describe a hyperbola in a continuous way, fix thumb tacks at the foci, F_1, F_2. Place the loop of a string whose ends are held together, over the tack F_1 and both strands around the other tack F_2. Fasten a pencil point to the loop at some point of the loop and draw the loop tight against the tacks. Then permit both strands to slide over F_2 together, keeping the string tight against the thumb tacks." A. R.

An Introduction to Coordinate Geometry and the Calculus. By N. J. Chignell and E. H. Fryer. Pp. vii, 231. 5s. 1933. (Blackie)

This book covers the field required for School Certificate with, in addition, a useful chapter on the differentiation of the circular functions. No previous knowledge of either of these branches of mathematics is assumed, the book being essentially a book for beginners. The bookwork is clear and concise and the examples, of which there are over a thousand, stimulating. Most of the early examples are numerical, but towards the end there are a number of exercises of a more theoretical type.

The two subjects are interwoven into a comprehensive course (as they should be), and the interest is never allowed to flag. Our main criticism is that the idea of an integral as the limit of a sum is nowhere introduced. This is in our opinion a serious mistake: boys take to the idea quite readily and it should be introduced early. It is important not only as a mathematical concept, but also because it greatly simplifies the application of integration to mechanics. Moreover, it forms a useful approach to approximate integration. This omission, however, should not be allowed to weigh too heavily, as it can easily be remedied by the teacher, and, apart from this, the book should prove to be most useful. Chapter VIII, giving in a short space some of the more important theorems on conic sections, is a valuable one. The printing is clear and the general appearance of the book attractive. N. R. C. D.

The Theory of Canonical Matrices. By H. W. Turnbull and A. C. Aitken. Pp. xiii, 192. 17s. 6d. 1932. (Blackie)

The *Gazette* notice of this book is, unfortunately, some two years late, and the worth of the book is well established before this short review of it is published. The delay is not the fault of the Editor nor of the present writer.

The book contains an excellent account of certain aspects of matrix theory. It is an ideal book for those readers who have already a fair knowledge of the elements of matrix theory and wish to know more: the elements are given here, but the book demands a readiness to think about matrices in a way that cannot be acquired by the learning of the matter contained in the introductory chapter. The appropriate parts of Professor Turnbull's earlier book (reviewed on page 466 of Volume XIV) form a suitable preparation for the beginner. W. L. F.

Post-Primary Mathematics. By W. F. F. Shearcroft and G. W. Spriggs. I. Pp. 172. II. Pp. 207. III. Pp. 213. 2s. 6d. each; without answers, 2s. each. 1934. (Harrap)

For the great majority of English children school life ends very early. But there is no good reason why in the last three years of that period they should not learn some mathematics, and these books show how much mathematics can be learnt by a child who leaves school at fourteen. The word "mathematics" is used advisedly, for the authors pay no attention to strict, impassable frontiers between the various mathematical subjects.

It is not, however, merely in this way that the authors obey Professor Whitehead's injunction not to "divide the seamless coat of education".

They go deeper and endeavour so to enlist the whole of the child's nature, that the act of learning shall become, in Professor Whitehead's words, a real "occasion of experience". To do this they use, for great part of the course, a story of a combined holiday visit to the sea-side, for which they provide a map. Questions of cost lead to examination of the ordinary "simple" and "compound" rules of arithmetic, already known on trust by the children. Now, however, these rules are rationalised and, together with the rules for decimals, are brought into one system. Then the formula is introduced and measurements on the map and details of the construction of tents bring in angles and areas and the beginnings of geometry. This for the first of the books; but the aim all through is to make each new development of the study a satisfaction of some felt need. In this way, at the end of the third year the children will have acquired a useful knowledge of all the valuable parts of arithmetic, some geometry and algebra—they will be able to solve quadratics—, an ability to use logarithms and an acquaintance with the nature and use of the three chief trigonometrical ratios.

These books can therefore be recommended to presidents of boards of education, education committees, inspectors, schoolmasters, schoolmistresses—whoever the people are who control our elementary education. But one thing must be said. The books are of no value without their story, their talk, their discussion. Now it is quite impossible to imagine children reading all that; except for an odd boy or two here and there they will not read it,—but they will listen to it. Thus the immediate appeal of these books is to the teachers. It is only through the teachers that the authors will be able to get at the children.

Detailed criticism might reveal faults in these books, but that is not necessary here. The books contain many examples for practice, but not too many; indeed, the last third of the third book is made up of examples. Perhaps some of the algebra there goes beyond that dealt with in the text. On the other hand, many of the geometrical problems are so devised as to add to the store of geometrical knowledge previously acquired. In geometry little stress is laid on formal proof, but many of the problems should lure the child on, not merely to accurate thinking, but to record of it in the written word.

The authors introduce the idea of a function and in one place give an account of the word which would be very puzzling to a child who read it. The word indeed is a puzzle. Functions are endless in number and variety; surely the elusive "x" is the most over-burdened official in the universe.

T. M. A. C.

Théorie mathématique de l'Assurance maladie. By H. GALBRUN. Pp. viii, 218. 60 fr. 1934. (Gauthier-Villars)

This work forms the sixth section of Volume III of the *Traité du Calcul des Probabilités*; the fourth and fifth sections have been reviewed (*Gazette*, XVII, July, pp. 221-3; December, pp. 335-7). The author here deals with a mathematical theory of Assurance in case of sickness, and follows the same method as that adopted in the previous sections, of considering the adventures of a single individual and expressing the probabilities as definite integrals. Thus, for example, he supposes the case of a life aged x, who has belonged to a class B, since age ξ, cured of his last illness, experiencing μ illnesses, cured of the last at age α within the interval (x, y), then surviving to age y in the class A, after having experienced ν new illnesses of which he is cured and so on. In this way the author builds up a mathematical theory and produces numerous formulae to express some of the possible happenings; 356 of these formulae are presented in the text.

Dr. Galbrun has given us here a very clear account of the way in which he has considered the subject, and has presented us with many formulae which might possibly be of use, but such a method does not appeal to one as being

of practical value. The author does indeed admit that the necessary statistics are not available, but says in his preface that nothing permits us to affirm that we shall never have surmounted this. Of course not, and it would be foolish to make such an unnecessary prophecy, but we venture to think that the great deviations, due to countless disturbances, would make such tables, if they could be constructed, of little value by the time they were completed.

There is, however, one point upon which it seems advisable to say a few words. The number of pages given up to life insurance questions is now about 1200 and the total cost of the five sections is 230 fr. When the great *Traité* was first announced, many students of "probability" must have hoped to procure the whole work, and must now feel this to be beyond their powers. Life is far too short, especially when some of it is taken up with $(\mu+\nu+1)$ illnesses, for us to prepare extensive formulae in advance on the chance that they might be useful to some future age. Surely at the present time the words given by Seneca nearly 2000 years ago should be kept in memory:

"nunc quae dementia est supervacua discere in tanta temporis egestate".

W. S.

Easily Interpolated Trigonometric Tables with Non-interpolating Logs, Cologs and Antilogs. By F. W. Johnson. $3.50. 1933. (Simplified Series Publishing Co., San Francisco)

The interesting part of this collection of five-place tables is the non-interpolating portion, which was described at length in the *Gazette* (Vol. XVI, p. 54) on its first appearance. The trigonometric tables give sines, tangents, cotangents, and cosines, natural and tabular-logarithmic, from 0° to 45° at an interval of 1′; each column is differenced, and a table of proportional parts for interpolation to 1″ is printed at each opening. A thumb-index exposes the contents of each individual page, and simple explanations form an introduction (44 pp.) which, with the contents and preface (x), occupies the middle of the volume. To add to the utility of the tables, a 5 in the last place is printed 5 or $\not 5$ according as the preceding digit should or should not be increased if four places only are required; if the 5 actually terminates the entry, it is marked for omission if the preceding digit is even, a common precaution against systematic error being thus applied automatically.

E. H. N.

Grundlagen der Mathematik. I. By D. Hilbert and P. Bernays. Pp. xii, 471. RM. 36; geb. RM. 37.80. 1934. Grundlehren der math. Wissenschaften, 40. (Springer, Berlin)

This is the long-awaited account of Hilbert's metamathematics, or, as it is now called, the *Beweistheorie*. The publication has been delayed to some extent by the necessity of taking into account the recent results of Gödel, and as a consequence the book is issued in two parts, the first and longest of which is the subject of the present review.

Hilbert's method has been several times mentioned in the *Gazette*. It partly consists in transferring the formalist view, now usual in geometry, to logic and arithmetic. If we take any logical scheme, as for example that in the *Principia*, we can empty the symbols of meaning and treat the whole theory as a game played in accordance with certain rules, and we may treat Peano's foundation of arithmetic in the same way. We can then make certain statements about games that can be played; for instance, we might be able to prove that no two games could end in combinations of symbols which, when interpreted, represent a proposition and its contradictory. We should then say our system was self-consistent.

In reasoning about the possible games we certainly use a logic, but it is a very different logic from the logic of general propositions; it is like the logic which is applied when we follow the moves of a game of chess and certify them as correct, or like the logic employed in verifying an identity in ele-

mentary algebra. The logic is applied to concrete objects, chessmen or recognizable marks on paper; it resembles the logic used by the geometers of antiquity when they reasoned on the actual figure lying before them and used those properties of it which were evident to the senses. Perhaps the main difficulty in understanding Hilbert's method arises from the fact that readers who have previously grasped the formalist view do not always grasp at once that in reasoning about the possibilities of formal proofs a return must be made to the earlier standpoint of thinking about concrete objects.

A kind of arithmetic is also used, but this is like the kindergarten arithmetic, which consists only in noting statements such as that if we compare the symbols ||| and ||||, there is an extra stroke in the latter symbol. Nothing like the Axiom of Induction is used in the reasoning, though the reasoning may be about the symbolic expression of that axiom considered as a mere array of symbols.

An obvious question must often have occurred to those who have reflected on mathematical proofs. The fundamental rules and laws of logic and arithmetic have been formalised. Thus to prove any theorem, *e.g.* Goldbach's conjecture, all that is apparently needed is to move the symbols about until some combination like $2N \subset Np+Np$ is reached, and yet of course the question must be immensely more difficult than this suggestion implies. This book goes a long way towards removing the mystery. Essential to the founding of arithmetic are not only Peano's axioms, but also his definitions of addition and multiplication by induction; it appears that it is the definition of multiplication which complicates the issue and prevents a theorem from being proved by a refined edition of the operation of counting.

After an introductory portion, the book begins with the calculus of propositions. In the rest of the book, this part of logic is used merely as a set of rules for manipulating combinations of the fundamental logical symbols, but an account is inserted, more complete than is to be found elsewhere, of the postulational foundation of the logic of propositions, and of the independence and inter-relatedness of the postulates.

Then comes the calculus of propositional functions, with the rules for treating expressions like $(x) f(x)$ and $(Ex) f(x)$, and complicated combinations of these and of the elementary logical symbols. After the symbol for identity is adjoined and its rules elaborated, it is proved, in a long section, that no contradiction arises when a combination of symbols is introduced which, if interpreted, would mean that the domain of objects considered contained an infinite number of objects. This leads to a discussion of the axioms of arithmetic. The next portion is a deep analysis of definitions by induction, and finally the notion of "the x such that", or rather the corresponding combination of symbols, is investigated.

Besides investigations on consistency, a main problem of the theory is the decision-problem (*Entscheidungsproblem*): if the symbols be interpreted, the problem is the same as that of finding a method which will decide whether any given theorem can or cannot be proved from given axioms. The solution is given here for certain cases in logic and for certain axioms of arithmetic. If the problem could be solved for the axioms of logic and arithmetic, together with the definitions by induction, the whole of the theory of numbers would be trivialised. This is as good as saying that the solution is impossible in general, and some idea of the dividing line can be gathered from this book.

The literature on this subject is very difficult to read, the proofs here are much simpler than those in the original papers, and there is any amount of new matter. One curious feature of the presentation is that the text is not broken up into sections and displayed theorems as is customary in mathematical works; this might perhaps have helped the reader a little in the longer proofs, one of which takes up 34 pages of close reasoning. There are very few misprints; on p. 110 the first implication-sign should be an equivalence,

there is a comma instead of a stop in a formula on p. 324, occasionally in complicated formulae there is an unclosed or loose bracket, and it would have been an advantage in these to have used Peano's device of dots for brackets.

Anyone who reads this book will hope that he will live long enough to see the sequel. It is easily the most important contribution to the question since the *Principia*. The subject is not yet a garden, but it is no longer a wilderness.

H. G. F.

Synthetische Geometrie. By H. Liebmann. Pp. viii, 119. RM. 5.60. 1934. Teubners mathematische Leitfäden, 40. (Teubner)

There are now many books, large and small, on projective geometry, but this contains some new things and a fresh treatment of some old matters such as Poncelet's porism, and those already familiar with the subject will find the book worth attention because of the treatment of the planar axioms.

If six points 1, 2, ... , 6 are joined in order to form a hexagon, and it happens that the cuts of opposite joins lie on a line, the hexagon may be called a Pascal hexagon. The author considers two axioms V_1 and V_2, of which the first asserts that if 1 ... 6 is a Pascal hexagon, so is the hexagon obtained when two odd points or two even points are interchanged, and the second asserts that this is the case when an odd point and an even point are interchanged. He shows that, in the presence of the fundamental axioms of association in the plane, V_1 and Desargues' theorem on perspective triangles are equivalent, and that V_2 implies Pappus' theorem. This rounds off very neatly the axiomatic introduction to the plane geometry.

The treatment of pencils of conics and the definition of metric on a projective basis are also worth notice. The book ends with an introduction to quadrics.

H. G. F.

Analytical Geometry of Three Dimensions. By D. M. Y. Sommerville. Pp. xvi, 416. 18s. 1934. (Cambridge)

This text-book, which was published in February 1934 only a few days after the death of the author, is a reminder to the mathematical world how great a loss has been sustained. Of the noteworthy books which Sommerville has written, the *Elements of Non-Euclidean Geometry* (1914), the *Analytical Conics* (1924), the *Introduction to the Geometry of* N *Dimensions* (1929), and the present volume, this last is not only the most mature but is also perhaps the most important.

It offers a course starting at the beginning of analytical solid geometry suitable for honours students in the universities. Everyone with any experience of college teaching is aware of two main problems which confront an author in this subject—the preservation of an effective geometrical treatment throughout the algebraic manipulation, and the choice of interesting and practicable examples. It is evident from the outset that Sommerville has had these two matters prominently before him, and has therefore laid the foundations of a thoroughly successful text-book. Modestly he claims that the book is partly to be regarded as an introduction to the inspiring volumes of Professor H. F. Baker on the *Principles of Geometry*: and there is every evidence that this will be the case.

The opening two chapters deal in traditional way with the rudiments, coordinates, the straight line and the plane. Alongside the Cartesian coordinates prominence is given to the vector and the matrix notation: and the early use of Plücker line-coordinates is welcome. Chapter III, which discusses general homogeneous coordinates, is widely different. As the author implies in the preface it is not written for the beginner. It aims at justifying the use of coordinates without the assumption of metrical properties. The one-to-one correspondence, cross-ratios, involutions, and the absolute of two and three dimensions are considered, together with the corresponding algebra of the

bilinear form and of the logarithm of a cross-ratio, so that the projective and metrical notions are brought into mutual relation. As a synopsis of an important part of algebraic geometry all this is good: but whether it will be intelligible to a reader without considerable further explanation is doubtful. Probably the only satisfactory method of laying these foundations unconfusedly is the axiomatic. The next three chapters, IV, V and VI, discuss the sphere, the cone and cylinder, and the remaining quadric surfaces respectively: Chapter VII contains further properties, Chapter VIII deals with the reduction of the general equation of the second degree, the treatment being eminently geometrical, and such concepts as polarity, projective and orthogonal invariance, the duality of null systems, the significance of the rank of a matrix are set forth to prepare the way for the actual reduction. This is duly undertaken with reference to the behaviour of the surface on the plane at infinity, and with careful attention here as always to the reality of the figure.

Chapter IX discusses generators and parametric representation, X plane sections, XI tangential equations, XII foci and confocal surfaces, XIII linear systems of quadrics. Chapter XIV introduces the parametric treatment of curves and developables, but expressly excludes differential methods. In XV the invariants of two quadrics are considered. In XIII and XV a complete statement of the projectively distinct relations of two quadrics is given, accompanied by an appropriate reference to the algebraic work on invariant factors. Chapter XVI is a useful introduction to line geometry with an appeal to the use of four and higher dimensions. Finally, Chapter XVII discusses algebraic surfaces, the Hessian, and more particularly cubic and quartic surfaces.

The prominent feature in the above programme is the welcome emphasis on the geometrical figure, particularly on the real and not imaginary figures, and the consequent subservience of algebra and of all heavy computation.

The format of the book is very pleasing and is a plain evidence of the care bestowed upon its production by author and printer alike. There is no doubt that it will fulfil a most useful purpose in giving to many an honours student an authoritative and effective introduction to present-day algebraic geometry.

H. W. Turnbull.

Inequalities. By G. H. Hardy, J. E. Littlewood and G. Pólya. Pp. xii, 314. 16s. 1934. (Cambridge)

A treatise by three of the leading mathematicians of the day on a subject of which we all learnt something at school should attract the notice of every mathematician. There are treatises on almost every conceivable mathematical subject, but apparently never before has one been written simply on inequalities. Yet there are many more or less simple inequalities which, as the authors say, are "of daily use" in analysis.

This book was originally planned as one of the series of Cambridge Tracts, but the authors soon found that they needed more scope than a tract would give them. The book contains ten chapters, of which the first is introductory and explanatory. Chapters 2-6 contain a thorough and systematic discussion of the inequalities "in daily use", and centre round the theorem of the arithmetic and geometric means, Hölder's inequality, and Minkowski's inequality. The last four chapters contain an account of more recent work in which the authors themselves have played a large part. Altogether there are 405 theorems, each embodying an inequality or a statement immediately connected with inequalities.

A large number of the theorems in the first part are quite easy to prove if we do not restrict our methods, or bother about refinements such as enumerating cases in which $\leqq$ becomes $=$. But the authors set themselves to be not merely "rigorous" and "refined" but "appropriate". Suppose, for example, that we want to prove Hölder's inequality

$$\Sigma ab \leqslant (\Sigma a^k)^{1/k} (\Sigma b^{k'})^{1/k'},$$

where $k' = k/(k-1)$, $k > 1$, in the case where k is rational and the number of terms is finite. This is a theorem of finite algebra; so, according to the authors' principles, there should be at least one proof which is purely algebraic, and does not depend on other methods such as the differential calculus, or even on ideas of continuity familiar in the theory of functions of a real variable. If k is irrational, a^k is not algebraic. It is defined as $e^{k \log a}$, and the theory of the exponential and logarithmic functions must underlie the proof.

The theorem of the arithmetic and geometric means provides a good illustration of this point. The inequality is

$$(a_1 a_2 \dots a_n)^{1/n} \leqslant \frac{1}{n}(a_1 + a_2 + \dots a_n).$$

The oldest and most familiar proof, due to Maclaurin, depends on the observation that, if two unequal numbers a_1 and a_2 are each replaced by $\frac{1}{2}(a_1 + a_2)$, the left-hand side is increased, while the right-hand side remains the same. The two sides are equal when all the a are equal. It follows that, if the ratio of the geometric to the arithmetic mean has a maximum which it attains for a definite set of a, the maximum must be unity; and the theorem follows. The existence of an attained maximum is familiar in the theory of continuous functions of a real variable, but it is not a matter of finite algebra. Accordingly the authors give Maclaurin's proof, but it is only their third proof of the theorem. Their first proof is one, due to Cauchy, in which we proceed from $n=2$ to $n=2^m$, and then backwards to other values of n.

Hölder's inequality illustrates another point. Suppose that we can prove it for rational k, with $<$ unless the a^k are proportional to the $b^{k'}$. We can deduce the irrational case by a passage to the limit ($k \to$ irrational limit through rational values), but the $<$ is lost in the process, and gives only $\leqq$ in the limit. But really it is still $<$ unless the a^k and $b^{k'}$ are proportional, and so we have failed to prove the complete theorem. Another method must therefore be sought.

Everyone will be able to appreciate the enormous labour involved in dealing with hundreds of inequalities in this spirit. I need hardly say that the authors emerge triumphant. It is not altogether easy going, and the beginner will still be glad of his ordinary algebra book with its single chapter on inequalities. But there will never be any excuse for any author making a mess of inequalities again.

What queer scraps of information one used to get out of some of the older books. I was brought up on C. Smith, and I have just looked up his chapter again. Only one of his theorems is anything but a variant of the theorem of the arithmetic and geometric means (of course proved by Maclaurin's method). The exception is the inequality

$$\mathfrak{M}_r^r \mathfrak{M}_{m-r}^{m-r} \leqslant \mathfrak{M}_m^m \quad (0<r<m),$$

where
$$\mathfrak{M}_r = \left(\frac{1}{n}\sum_{r=1}^{n} a^r\right)^{1/r};$$

and this is a "weak" inequality in the sense that it is an immediate consequence of the simpler one

$$\mathfrak{M}_r \leqslant \mathfrak{M}_s \quad (r<s),$$

(H.L.P. Theorem 16), which Smith does not give. Chrystal does at any rate manage this. But it is only when the fundamental importance of Hölder's and Minkowski's inequalities is recognized that the subject attains any sort of unity.

The second part of the book deals with applications of the calculus of variations, bilinear and multilinear forms, Hilbert's inequality and its extensions,

and rearrangements. These chapters are a series of essays on subjects in which the authors themselves have been interested, and do not aim at the completeness or inevitability of the first part. They are intensely interesting to the analyst, and give in a very compact form the results of one of the most successful of recent lines of research.

The theorem that if Σa_n^2 and Σb_n^2 are convergent, then so is the double series

$$\Sigma\Sigma \frac{a_m b_n}{m+n},$$

was discovered by Hilbert, and the first elementary proof of it was given by F. Wiener in 1910. But it was a long time before a really simple proof was discovered, and the simplification of the proof was the starting-point of a great deal of research. In a most entertaining lecture Hardy gave nine different proofs of the theorem, each depending on some ingenious device. The subject was a prominent feature of the early volumes of the *Journal of the London Mathematical Society*, which started in 1926. The present volume was foreshadowed in Hardy's presidential address to the society in 1928, entitled "Prolegomena to a chapter on inequalities". The address, published in Vol. IV of the *Journal*, should be re-read with the finished volume in hand.

It was the subject of rearrangements which first brought the Hardy-Littlewood-Pólya syndicate into action. They considered the question of what arrangement of the x and y makes

$$\sum_{r=1}^{n} \sum_{s=1}^{n} a_{r-s} x_r y_s$$

a maximum, if the a are given, $a_0 \geqq a_1 \geqq a_2 \geqq \dots \geqq 0$, $a_r = a_{-r}$, and the x and y are given in all but arrangement. Another problem of the type was explained by Hardy and Littlewood to the readers of *Acta Mathematica* in terms of cricket averages. Suppose a batsman plays, in a given season, a given stock of innings, $a_1, a_2 \dots a_n$. Let α_ν be his average after the νth innings, $\alpha_\nu = (a_1 + a_2 + \dots a_\nu)/\nu$. Let $s(x)$ be a positive increasing function, and let his "satisfaction" after the νth innings be $s(\alpha_\nu)$. Let his total satisfaction for the season be $\Sigma s(\alpha_\nu)$. It is a rather paradoxical result that his satisfaction for the season is greatest if his innings are steadily decreasing. The proof of this is easy enough. Suppose, however, we take, instead of α_ν, the maximum average for any series of innnings ending at the νth,

$$\operatorname*{Max}_{\mu \leqq \nu} \frac{a_\mu + \dots + a_\nu}{\nu - \mu + 1}.$$

Then the same result still holds. This theorem, which has important applications in the theory of functions, is far from easy, and the extremely elegant proof of its integral analogue given in § 10. 19 is the end of a steadily decreasing sequence of proofs.

I hope I have said enough in the way of expressing my admiration for this book. It will immediately take its place as part of the essential equipment of every analyst. A good student should be able to manage much of the first part, and there is a large number of examples.

I have noticed only one mistake: in Theorem 281, $\int_0^\infty f^r\,dx$ should be raised to the power $1/r$. I do not know whether it is a mistake or an innovation to write "nul" for "null". The *Shorter Oxford English Dictionary* gives "null" only, but I expect things are different in Cambridge. E. C. TITCHMARSH.

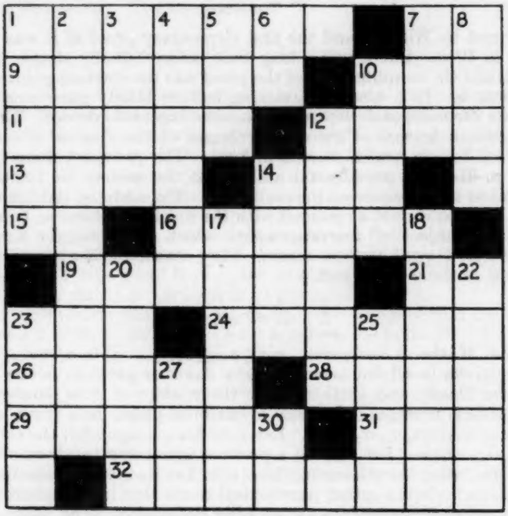
Abstracts of Dissertations for the degree of Doctor of Philosophy. VI. Prepared by the Committee for Advanced Studies, University of Oxford. Pp. v, 303. 3s. 6d. 1934. (Oxford ; at the Clarendon Press)

These abstracts are classified into eight main groups, and my first impression

was that there was nothing of mathematical interest in the volume. But a more patient search revealed that dissertations on Hankel transforms, on the Riemann zeta-function and on the Epstein zeta-function are considered to be contributions to the study of the " Physical Sciences ". T. A. A. B.

---

994. CROSS-WORD.



Clues across.

1. Pounds in a metric ton.
7. Only one cake had more than $2x^2$ candles.
9. π.
10. Miraculous draught.
11. $\sqrt{2300}$.
12. Multiple of 9.
13. (Rev.) $\frac{1}{6}$ of 12 across.
14. Sum of two squares.
15. Twice a square.
16. A cube.
19. A cube.
21. A square.
23. Sum of squares of two consecutive numbers.
24. Inches in 10 Km.
26. Multiple of 127 and of 19.
28. Feet in Km. $\div$ 30 down.
29. π times a tenth power.
31. Multiple of 43.
32. A 13th power.

Clues down.

1. Square of 10 across.
2. e.
3. (Rev.) 28 across $\div$ 10.
4. 7 down + a square.
5. Decimal of a cubic foot in a gallon.
6. A square + 2.
7. A square.
8. A square.
10. Baked in $\sqrt{(19 \text{ across})}\,\pi$.
12. Multiple of 6001.
14. Multiple of 79.
17. Same as 24 across.
18. $\sqrt[3]{2200000}$.
20. 15 across $\times\, 10^6 H(A+M)$.
22. Cubic inches in a Kl.
23. 5 times a fifth power.
25. 7 times a tenth power.
27. Souls in Paul's ship − longest Psalm.
30. Spider's legs $\times$ Ezra's chapters.

BETTER LATE THAN NEVER.

An editor is often said to command the services of reviewers. No editor who attempted to do anything of the sort would long enjoy such help as has established the reputation of the *Gazette* in this field. The friendly method has, however, its disadvantages: it is more sensitive than a mere routine to interruption by illness, and now and again a promise does not materialise. Thus a book which ought to be reviewed is not brought to the notice of readers.

The following is an account of most of the books of which receipt has been acknowledged during the years 1921-1930 and nothing more has been said. The notes that have been added, while less ample, are not less serious than timely reviews would have been, and it is hoped that they will be useful to readers and will increase the confidence of authors and of publishers in the *Gazette*.

Notions Sommaires de Géométrie Projective à l'usage des candidats à l'Ecole Polytechnique. By M. d'Ocagne. Pp. vi, 26. 3 fr. 1924. (Gauthier-Villars)

By concentrating on essentials and on a few lines of development, this clearly written summary does succeed in conveying a sense of the power of projective methods. The foundation is of course metrical. The applications are to the theorems of Desargues and Pascal and to the theory of poles and polars, the conic being recognised as generated by the intersection of corresponding rays in homographic pencils. There are paragraphs on homography in space and on conicoids, but although the author's first section is on imaginary elements, the ordinary reader will hardly guess that the arguments refer to ellipsoids as well as to the conicoids he knows to be ruled. For students who are to have a thorough course in the subject, to skim cream in advance is questionable policy, but for others, the pamphlet may be recommended warmly.

Leçons sur les nombres transfinis. By W. Sierpinski. Pp. vi, 240. 40 fr. 1928. Collection Borel. (Gauthier-Villars)

The Polish school of mathematicians has become very prominent during the last ten or twelve years, mainly because of its magnificent work on all points connected with the theory of functions of a real variable. In particular, the journal *Fundamenta Mathematicae* is devoted entirely to the general theory of aggregates; and in the foundation and conduct of this periodical a leading part has been played by Professor Sierpinski. His account of the transfinite numbers was therefore certain to be interesting and authoritative, and forms a notable addition to the well-known *Collection Borel*.

The book opens with a discussion of the idea of an aggregate or "ensemble": transfinite cardinals occupy the first 140 pages, transfinite ordinals the remainder. The notorious axiom of Zermelo has its due share of attention. Much of the material can be found in various treatises, but we are considerably indebted to the author for his work in writing a self-contained account of a very important subject. The English reader has at his disposal Professor J. E. Littlewood's masterly *Elements of the Theory of Real Functions* as an introduction to more massive works, but this is severe reading when divorced from the author's lecture-room commentary, so that Professor Sierpinski's more discursive volume makes a valuable addition, and is sure of a place on the shelves of all who are interested in the foundations of mathematical analysis.

L'Analysis Situs et la Géométrie algébrique. By S. Lefschetz. Pp. vi, 154. 20 fr. 1924. Collection Borel. (Gauthier-Villars)

The appearance of this Borel tract is the most important landmark in the history of algebraic geometry since the war. While topology and function-theory went hand in hand in the study of algebraic functions of a single variable, the connections between their generalisations to constructs of higher

dimensions were of the flimsiest, prior to the investigations of Lefschetz. Lefschetz changed all that, and the tract under review has been the means of bringing his superb contribution to the notice of the general mathematical world. It is therefore all the more regrettable that the monograph should contain so many typographical errors of an annoying nature, which make it unduly difficult to read. In addition to this, it must be confessed that some of the proofs are far from convincing. This drawback is, however, not so serious, as a competent reader can usually remedy these defects easily, and they can be forgiven in one who has opened up such a vast field for profitable and fascinating research.

The tract is devoted to laying the foundations for the applications of topology to algebraic geometry (which of course includes algebraic functions). Much space is therefore devoted to the study of the Riemannian manifold of an algebraic surface, and the writer of this notice must confess to experiencing a thrill when he first saw before him the natural explanation, in terms of topology, of certain invariants of a surface which had hitherto appeared somewhat obscure. A chapter is devoted to Poincaré's investigations of the Abelian sums, and it is particularly interesting to see how Severi's theory of the base for algebraic curves on a surface can be explained in terms of simple topological ideas. Subsequent chapters deal with the extensions to algebraic varieties of higher dimensions, and to the theory of Abelian functions. An appendix shows how the notions of topology enable us to give a very concise treatment of the double integrals of the second kind attached to a surface.

Leçons sur quelques Équations Fonctionnelles. By E. Picard. Pp. 184. 40 fr. 1928. (Gauthier-Villars)

M. Picard has renounced his intention of writing a fourth volume of his *Traité d'Analyse*; but he has so far relented as to accede to requests from various pupils to publish in book form some of the courses of lectures which he has delivered at the Sorbonne during the last twenty years. We are very grateful to those who made the requests, and to M. Picard for meeting their wishes; for of all our great mathematicians there are few who can write so delightfully as M. Picard.

The present volume is divided into four chapters. In the first, equations of the type $f(x+y)+f(x-y)=2f(x)f(y)$ in the real domain are considered, and a pleasant application to non-euclidean geometry is added. In the second chapter, the author considers various special functional equations in the complex domain: there are interesting sections on the gamma function, elliptic functions and the transcendental functions usually named after Poincaré. Difference equations form the text of the third chapter; the point of view of the theory of functions is adopted, so that relations are established with Jacobi's Θ-function, Lamé's equation and certain transcendental functions which were first investigated by Picard himself in two papers in the *Acta Mathematica*. The last chapter studies Abel's functional equation

$$f[\theta(z)]=f(z)+1,$$

where $\theta(z)$ is a given function, $f(z)$ the unknown function; there is an application to Fredholm's integral equation.

The interesting nature of the subject matter, expounded in M. Picard's superb style, makes this volume most fascinating.

Leçons sur les Conduites. By C. Camichel. Pp. 102. 30 fr. 1930. (Gauthier-Villars)

The technical researches of the past twenty years on the flow of liquids through pipes are described briefly in this interesting book. In the first part the theory of certain phenomena in the flow of liquids is dealt with, such as disturbances due to air pockets, irregularities in the walls of the pipe, sudden opening and closing of the pipe. The question of resonance due to such dis-

turbances is discussed, and pipes of uniform and non-uniform section and thickness are dealt with in separate chapters. The second part of the book is concerned with the bearing of the results contained in the first part on the actual construction of water pipes, and consideration is given to fractures in pipes and other accidental disturbances.

The work, which is well written, is primarily of interest to the hydraulic engineer, but contains some interesting methods and analysis.

Leçons sur la Théorie des Tourbillons. By H. Villat. Pp. 300. 65 fr. 1930. (Gauthier-Villars)

This is an excellent book, based on a series of lectures given in the Paris Faculty of Sciences in 1929. The theory of vortices is developed in some detail, starting with the Eulerian equations of motion for a fluid. After obtaining the usual formulae for plane motion, an account of the Bénard-Karman vortices is given, with its application to the calculation of the resistance to the motion of a solid in an infinite expanse of liquid, followed by a consideration of motion in a liquid with finite boundaries. The work of Synge, Kirchoff, Hill, Korn and others is described, and there is a chapter on the general theorem of Lichtenstein. The author concludes with a chapter on vortices in viscous liquids. The more important recent developments are dealt with briefly. The book is written with a clarity which one expects from French authors, and at the same time the practical aspect of the subject is not lost sight of.

Essai Philosophique sur les Probabilités. By P. S. Laplace. I. Pp. xii, 104 ; II. Pp. iv, 108. 3 fr. each. 1921. Les Maîtres de la Pensée Scientifique, 10. (Gauthier-Villars)

A convenient issue of this classical essay, which was expanded from a lecture given at the École Normale in 1795 to be the introduction to the second edition of the *Théorie Analytique des Probabilités*. For a critical analysis, see Todhunter, pp. 497-504. In this new edition the print is delightful, but the paper is unfortunately of the poor quality to which French publishers were reduced ten years ago. Otherwise the only matter for regret is that it was not possible to produce in this case one double volume of the series, as is often done in the Manueli Hoepli.

Exposé élémentaire du Calcul Vectoriel et de quelques Applications. By H. Malet. Pp. viii, 74. 1927. (Gauthier-Villars)

The definitions are cartesian, but the intrinsic character and consequent invariance of the several functions are recognised. The applications to hydrodynamics and electrodynamics are lucidly described. Why the familiar formulae for dissection of a vector field into its lamellar and solenoidal components are thought by M. d'Ocagne to be *un peu oubliées*, it is hard to guess, but it is true that they are often associated with Helmholz' name whereas M. Malet ascribes them to Vaschy ; it is Vaschy who is forgotten.

Leçons d'Analyse Fonctionnelle. By P. Lévy. Pp. vi, 442. 35 fr. 1922. Collection Borel. (Gauthier-Villars)

This is a valuable book, but as a whole it is out of place in the Collection Borel. The first part, *Les Fondements du Calcul Fonctionnel*, would have formed an admirable independent volume, designed, we might almost say, for the general reader. The second part, *Les Equations aux Dérivées Fonctionnelles du premier Ordre*, and the third part, *La Notion de Moyenne dans le Domaine Fonctionnel et l'Equation de Laplace généralisée*, are independent of each other, and would have been, in size and standard, normal monographs for the Collection.

No better introduction to the subject than the first part could be imagined, unless indeed it is Prof. Lévy's own smaller introduction in the *Mémorial* series. Continuity, integrability, first and second variations, homogeneous functionals of different degrees, the geometry of functional space, and integral equations, are all treated carefully and attractively.

The second part gives an account of results due partly to the author himself, and partly to R. Gateaux, who was killed in the first months of the war. The adaptation of the fundamental ideas of the theory of partial differential equations of the first order is remarkably complete.

We appreciate the achievement of constructing a satisfactory theory of mean values in functional space when we learn that the volume of a sphere in this space is almost all concentrated near the surface of an equatorial belt, and that the measure of a volume in functional space is almost always zero or infinite! Yet not only is a form of Laplace's equation for functional space discovered, but Green's functions are introduced for its solution. It must be confessed, however, that this theory, unlike so much of the theory of functionals, is developed only for its own sake; it does not coordinate or illuminate problems in ordinary potential theory.

Alike in the most elementary and the most advanced parts of the book, Prof. Lévy writes with the easy lucidity of a Frenchman who is master of his subject. He prefers a recapitulation to a reference, and so renders his work as far as possible self-contained. The volume is to be recommended warmly, especially as we have nothing in English which covers the ground.

Tables Numériques des Équations de Lagrange. By N. Nielsen. Pp. xviii, 400. 1925. (Gyldendalske: Gauthier-Villars)

These are tables connected with the equation

$$u^2 - av^2 = \pm w,$$

in which a is not a perfect square and w is prime to a. The first two tables, occupying half the book, give solutions of the equation for all values of a from 2 to 102 and for a selection of greater values of a, and for all values of w not greater than 1000. The author remarks that it is not always the case that every solution can be found automatically from one primitive solution, and he gives as many solutions as are necessary for the complete solution of the equation; four primitive solutions are frequently required.

If w is of the form p^m, where p is a prime, there is a least value of m, called the height of p with respect to the base a, for which the equation possesses solutions. Most of the tables in the second part of the volume are tables giving for bases decomposable in certain ways the heights of primes not greater than 101.

Prof. Nielsen has not handled so large a body of material without gaining a deep insight into the mathematical theory of the equation, but his contributions to the theory are to be found elsewhere. The tables will be of great service to all workers in this field, and they will be welcomed also by the large number of people who, without claiming to be masters of diophantine analysis, take an interest in arithmetical identities. They are beautifully printed.

The Ideal Aim of Physical Science. By E. W. Hobson. Pp. 34. 2s. 1925. (Cambridge)

Professor Hobson's lecture, delivered before the University of London at King's College, attempts a brief description of some possible answers to the query: "What is the true function of the physical sciences?" He divides the answers into two main categories, sufficiently defined by the adjectives "explanatory" and "descriptive", and lends the weight of his authority definitely to the latter, more modern, point of view. The latter part of the lecture consists of illustrations provided by some modern physical theories.

A Treatise on the Theory of Bessel Functions. By G. N. Watson. Pp. viii, 804. 70s. 1922. (Cambridge)

Prof. Watson's treatise was a predestined classic on the day on which it appeared, though no one would have ventured to predict that it would be readable as well as authoritative. The absence of a review in the *Gazette* at the time is explained by a paragraph on the last page of vol. xi: "The Editor will be grateful if the member to whom he sent, on its publication, a copy . . . will kindly inform him of its whereabouts." But why Mr. Greenstreet failed to make the usual memoranda in this case remained a mystery, and no delinquent ever claimed the promised gratitude.

A bibliography of 38 pages, and 87 pages of numerical tables, many of them calculated for this volume, give some indication of the scale on which Prof. Watson planned his work and of the immense industry which he devoted to it. But ambition and industry, though not always combined with accuracy, are comparatively common. It is in the 650 pages of mathematical exposition that Prof. Watson's rarer qualities are revealed. An encyclopaedic wealth of knowledge of the subject and of whole realms of theory which it illustrates, infallibility in logic, lucidity in argument, all compel admiration. On the vexed question of the choice of canonical functions of the second kind, Prof. Watson writes fairly and persuasively, and it is to be hoped that for the sake of uniformity his decisions will be accepted in the future even by writers, if any there are, who are not convinced by what he says.

While the author's outlook is that of the pure mathematician, he has "endeavoured to include all formulae, whether general or special, which, although without theoretical interest, are likely to be required in practical applications".

About a third of the theoretical work is elementary, if this description is interpreted in a sense sufficiently generous to extend to the representations of the various functions by contour integrals. A short chapter deals comprehensively with asymptotic expansions in which the argument is large compared with the order of the function. When the order is large also, much more recondite considerations, to use a favourite phrase of the author's, are involved, and the long chapter on this subject is one of the best in the book. After chapters on certain associated polynomials and functions and on the so-called addition theorems, come chapters on definite integrals, finite and infinite. The author deprecates the length of the chapter on infinite integrals "in spite of its incompleteness"; this is perhaps a way of deprecating its incompleteness in spite of its length. Indeed, when we consider that Ramanujan's curious results in which integration is effected with respect to the order of the function can be spared less than a page, and that the theory of transforms was only germinating when this book was written, it seems likely that infinite integrals involving Bessel functions will some day form the subject of an independent treatise.

In applications of Bessel functions it is at once evident that the positions of the zeros are fundamental, and problems connected with the distribution of zeros, with their numerical evaluation, and with their dependence on the order of the function, are of interest equally to the mathematician and the physicist. These problems may be attacked in two ways, either by the application of a general theory of the behaviour of solutions of differential equations, or by a study of explicit expressions for the particular functions with which we are at the moment concerned; the one method is the simpler and the more instructive, the other leads to more precise results. Prof. Watson has chosen, except in one section, the latter method.

Similarly in dealing in four chapters with four important types of expansion in series, Prof. Watson relies very little on general theorems, but he discusses with meticulous care the possibilities of the four particular expansions. In his preface, Prof. Watson describes the Bessel functions as admirably adapted for applications of the fundamental processes of the theory of functions of complex

variables. Perhaps it is only natural that where the background does not belong to this theory, and would not be familiar to the average reader, he has preferred usually to ignore the background.

The reputation of Cambridge is steadily reviving; Prof. Watson with this volume plays a worthy and a magnificent part.

An Introductory Course of Mathematical Analysis. By C. Walmsley. Pp. x, 294. 15s. 1926. (Cambridge)

This course, based on the syllabus drawn up by Dr. W. H. Young when Professor at Aberystwyth, necessarily has individuality. It is recognised that, when the notions of a cut and of a bound are familiar, there is a definition of a^x, for any positive value of a and any real value of x, which is inevitable before the exponential theorem is known, and which brings with it a definition of $\log_a x$ independently of any problems of integration. The limits of oscillation of a general sequence $\{s_n\}$ are defined by means of monotonic sequences: the upper limit is the lower bound of the sequence whose members are the upper bounds of the several sequences

$$s_1, s_2, s_3, \ldots; \; s_2, s_3, s_4, \ldots; \; s_3, s_4, s_5, \ldots; \; \ldots; \; \ldots.$$

But if the book is in some respects provocative it is also provoking. The way to avoid the usual difficulties in the treatment of limits of a function of a continuous variable is open; as a function of the positive variable h, the upper bound of $f(x)$ for values of x between a and $a+h$ cannot increase as h decreases, and the lower bound of this function is the upper limit of $f(x)$ on the right at a. But the opportunity for consistency and simplicity is missed, or rather, is relegated to an example, and in the text is the definition, explained no better if no worse than in a dozen other books, by means of an arbitrary sequence of values of x. Also the author is too ready and too careless with his appeals to "easy" and "obvious". Consider, for example, a paragraph on the "estimate of error" of the series $1+\frac{1}{2}+(\frac{1}{2})^2/2+(\frac{1}{2})^3/3+\ldots$. Writing E_n for $s-s_n$, Mr. Walmsley says: "We have evidently that E_n is the sum of the infinite series beginning with the $(n+1)$th term of the original series, viz.

$$(\tfrac{1}{2})^n/n+(\tfrac{1}{2})^{n+1}/(n+1)+\ldots,$$

which is clearly the unique limit and upper bound of the sequence whose $(m+1)$th term is $(\frac{1}{2})^n/n+(\frac{1}{2})^{n+1}/(n+1)+\ldots+(\frac{1}{2})^{n+m}/(n+m)$, m being any positive integer". The "evidently" is an invitation not to think, at the precise stage where a student should be urged to refer every assertion about a sum back to the definitions; following this, the "clearly" marks the writer as a dangerous guide, for the phrase amounts to arguing that because E_n is a sum, therefore E_n satisfies the definition of a sum.

Mr. Walmsley has not the gift, so valuable for work of this kind, of breaking an argument down into short steps, and he can be convicted of mistakes in detail. Nevertheless, an independent teacher could use his book with profit in planning a course.

Atomicity and Quanta. By J. H. Jeans. Pp. 64. 2s. 6d. 1926. (Cambridge)

The Rouse Ball Lecture for 1925 should bring the paradoxes and the conclusions of the subject within the store of general knowledge of any reader to whom a mathematical formula is not intrinsically unintelligible.

Principia Mathematica. By A. N. Whitehead and B. Russell. Second edition. Vol. II, Pp. xxxii, 744. 45s. Vol. III, Pp. viii, 492. 25s. 1927. (Cambridge)

That these volumes are unaltered from the first edition, being in fact reproduced photographically, is due not to any claim of perfection, but to the intricate interdependence of the work. A single alteration near the beginning

of the *Principia* would have invalidated a hundred references, implicit or explicit. The authors have therefore explained in an introductory essay in the first volume, with which we are not here concerned, the nature of the differences there would have been had the work been recomposed.

Higher Mechanics. By SIR H. LAMB. Second edition. Pp. x, 292. 15s. 1929. (Cambridge)

The first edition of this well-known treatise was discussed in a characteristic review of Greenhill's (*see Gazette*, vol. x, p. 309). In this second edition the main changes are due to the rewriting of certain sections: the author is determined to have every point expressed with perfect clarity. The price is much more reasonable than the 27s. 6d. demanded for the first edition.

Solutions to the examples in " A Treatise on the Dynamics of a particle and rigid bodies ". By S. L. LONEY. Pp. 240. 17s. 6d. 1926. (Cambridge)

All Professor Loney's books are packed with examples, and a key to one of his most popular works is a useful possession for any student of dynamics with sufficient conscience to use the key reasonably.

Modern Astronomy: its Rise and Progress. By H. MACPHERSON. Pp. 196. 6s. 1926. (Oxford Univ. Press)

There is no mathematics in this readable account of astronomical achievements and problems, which is based on a course of public lectures and has no pretensions to completeness in any respect. About half of the book deals with the solar system, the other half with stellar distribution and evolution. The author has a pleasant way of stating arguments on both sides of a controversy.

Science and Civilisation. Essays arranged and edited by F. S. MARVIN. Pp. 350. 12s. 6d. 1923. (Oxford Univ. Press)

Based on twelve lectures given at the Sixth Unity History School at Woodbrooke in the previous summer. These included one by J. L. E. Dreyer on " Greek mathematics and astronomy ", and one by A. N. Whitehead, entitled " The first physical synthesis ", describing brilliantly the revolution in thought caused by Galileo and Newton.

The Development of the Sciences. Lectures at Yale University. Edited by L. L. WOODRUFF. Pp. xiv, 328. 16s. 1923. (Yale: Humphrey Milford)

The first chapter is a valiant attempt by E. W. Brown to summarise in forty-two pages the history of mathematics. Included among the portraits are those of forty-four mathematicians, physicists, and astronomers.

A Study of Mathematical Education. By B. BRANFORD. New (second) edition. Pp. xii, 432. 7s. 6d. 1921. (Clarendon Press)

For an account of the original edition, see vol. v, p. 105. " I have seldom read a book with the motives of which I feel in such sympathy, and with the conclusions of which I so fully agree as this one. Mr. Branford's work is an advocacy . . . of heuristic and historical . . . methods in mathematical education." In the preface to the new edition, the author expresses his encouragement by the world-wide welcome given to the work, and refers to translations into German and Russian.

There are slight alterations throughout, but the additional material is in a forty-page appendix, of which the most important sections are the first and the fifth. (1) It has been established, largely by enquiries instigated by the author, that a diminution in technical accuracy is a normal feature of adolescence; the reasons for this, and the bearing on the teacher's problems, are discussed with characteristic insight and sympathy. (5) " Since the dethronement of Euclid the geometrical school world has been in a state of unstable

equilibrium." Gains and losses are analysed at some length, and there is a strong plea that teachers should make themselves familiar with comparative geometry, and bear in mind always that to deal with Euclidean geometry only is as if one should study the parabola in detail and know nothing of the general conic. Reading this section we regret that Mr. Branford was not invited to join the Committee which drew up the M.A. Report of 1923, since the conclusions would have been substantially the same, while his experience would have enabled them to be reached more rapidly and his name would have added to the influence of the Report.

Algebraic Equations : an introduction to the theories of Lagrange and Galois. By E. Dehn. Pp. xii, 208. 21s. 1930. (Columbia University: Humphrey Milford)

Serret practically defines Algebra as that body of mathematical knowledge which concerns the resolution of algebraic equations. Up to the time of Lagrange, such knowledge was often vigorously sought, but nevertheless remained unsystematic, and showed few signs of developing into a unified and aesthetically-pleasing general theory. That such development took place is due very largely to the two men of genius whose names appear in the title of Dr. Dehn's book. Lagrange set himself the problem of discovering a general method for the solution of the general algebraic equation. The rare genius of Galois recognised as the central point of Lagrange's work the connection between the solution of an equation and the permutations of its roots, and applied the resources of the theory of groups to extend and, in a measure, to complete the work of Lagrange. This achievement of a boy scarcely out of his 'teens is one of the most amazing incidents in mathematical history.

Dr. Dehn has written an elementary and self-contained exposition of this domain. He assumes very little previous knowledge on the part of his readers, and develops the study of permutations and groups from the beginning. His style, if hardly polished, is lucid and efficient ; it is materially assisted by some excellent type-setting. Those who are beginning the study of Galois theory will find this book of considerable use : but it is regrettable that they should be expected to pay 21s. for a volume of 200 pages.

Transformations of Surfaces. By L. P. Eisenhart. Pp. x, 380. 20s. 1923. (Oxford Univ. Press)

As a branch of differential geometry, the subject of the transformation of surfaces, or in other words of point-to-point correspondences between surfaces, is apt to seem rather structureless. The topological problems involved are easy to appreciate, but usually there is nothing to be gained by attempting to handle these by means of curvilinear coordinates. On the other hand, if specific relations between the two quadratic forms that are fundamental on one surface and the two that are fundamental on another are to be suggested, the possibilities are endless. The classical problem of deformation was at least definite, but the emphasis of the theory of relativity has tended for some time to confine attention to properties which depend on the first quadratic form only, and from this point of view the problem disappears. Nevertheless, not merely is this problem genuine, but its investigation during half a century resulted in the study of a number of other transformations.

Given a point-to-point correspondence between two surfaces, there is in general just one pair of conjugate families of curves on the one surface such that the corresponding families on the other are conjugate there. Thus to regard the correspondence as a relation between the conjugate networks rather than between the unstriated surfaces is to gain in precision without losing in generality. Consider now the doubly infinite set of lines joining corresponding points of the two systems. This set can be stratified in two distinct ways as a singly infinite set of developable surfaces ; in other words, the lines compose

a pair of families of developables, each line belonging to one member of each family. The pair of families of developables cuts each of the original surfaces in a pair of families of curves. The pairs of families so traced are as a rule wholly different from the pairs determined by the condition of conjugacy on both surfaces, but if it should happen that the families of developables trace out the conjugate networks, then the two networks are in the special relation which Prof. Eisenhart describes as fundamental. His book shows how this conception brings into one theory many transformations which have been considered separately and connects these transformations with many problems to which their application is by no means obvious.

To be ignorant that if the curves of reference on a surface are conjugate, the coordinates of the current point satisfy a partial differential equation of a special kind, is not necessarily to be mathematically uneducated, and Prof. Eisenhart's book must therefore be said to be for the specialist. But the amount of special knowledge required for its understanding and enjoyment is remarkably small. Moreover, the subject matter becomes steadily more tangible as the subject develops. We begin in n dimensions with a conjugate network which must not be assumed to lie on a surface. Two-thirds of the way through the book we are dealing with pentaspherical coordinates and with the correspondence between spheres in three dimensions and lines in five dimensions. The last chapter is on surfaces applicable to quadrics.

Prof. Eisenhart's success in unifying his subject is as remarkable as the range from which the individual theorems are drawn. His style is clear, and although sometimes the interest or importance of an idea must be taken for a time on trust, perseverance is soon rewarded.

Riemannian Geometry. By L. P. Eisenhart. Pp. viii, 262. 13s. 6d. 1926. (Princeton: Oxford Univ. Press)

Non-Riemannian Geometry. By L. P. Eisenhart. Pp. viii, 184. $2.50. 1927. Colloquium Publications 8. (Amer. Math. Soc.: Bowes and Bowes)

In view of the time that has elapsed since their appearance, these books may be taken together, and this is the more natural, since the second was in the author's mind before the first was finished. There is no law of excluded middle in nomenclature. Geometry that is not euclidean is not necessarily non-euclidean, and non-riemannian, far from being a description of every geometry that is not riemannian, distinguishes a class of geometries of which riemannian geometry is itself a member.

The conception of a manifold as determined by a quadratic differential form for the square of the line element, made explicit by Riemann in 1854, was almost neglected for many years. That a measure of curvature could be assigned to a manifold at any point by means of internal measurements alone, seemed to a few philosophers of profound significance, and a few mathematicians discussed spaces of constant curvature. Then a series of papers by Ricci, from 1884 onwards, created the theory of quadratic differential forms. How long the pure mathematicians, left to themselves, might have suffered this subject to remain in comparative obscurity is hard to tell. Just as a knowledge of it was beginning to spread, Einstein found in it the technique he needed for developing the theory of relativity, and suddenly the subject became one which everyone must learn. In spite of Einstein's applications, it was less a branch of geometry than a branch of analysis in which a certain amount of geometrical language seemed appropriate, as indeed such language had long been recognised to be in dynamics. The change came in 1917, when Levi-Cività discovered how the notion of parallelism, previously supposed to be utterly inseparable from that of space as a whole, could be introduced into infinitesimal geometry: Riemannian geometry came into existence.

An ordinary curved surface has associated with each of its points a tangent plane, and the effect of the new conception of parallelism was to establish a

connection between directions in one tangent plane and directions in another. This connection has nothing to do with the relation of the planes in space, and the conception of a larger space containing the surface and the planes is superfluous; the surface is a two-dimensional manifold that is not a euclidean plane, and the tangent plane is a euclidean plane associated with a point of the surface. In this order of ideas an extension is immediate. The tangent space at a point of a riemannian manifold of n dimensions is a euclidean space of n dimensions associated with that point, but we are under no necessity to think of the family of tangent spaces as immersed together in a space of higher order. A two-dimensional manifold does not exist as a surface in three-dimensional euclidean space unless certain well-known equations are satisfied, but its tangent planes exist, and if changes in direction in the passage from one plane to another can be correlated with changes in position of the point of contact in the manifold, a theory of curvature can be worked out for the manifold.

The development of the new subject was rapid. For it is not only the internal properties of a surface which offer themselves for consideration. If a two-dimensional manifold is to exist as a surface in three-dimensional euclidean space, the flatness of the space imposes certain conditions on the curvatures of the manifold; this is the meaning of the Gauss and Codazzi-Mainardi equations. In exactly the same way, the curvature-system of any three-dimensional manifold imposes conditions on the curvatures of any manifold which it can contain. Thus the theory of curvature of a surface in space, as well as the theory of geodesic curvature of curves on the surface, was waiting to be extended, not merely to two dimensions in relation to three but to m dimensions in relation to n, and mathematicians had ample material with which to elaborate a satisfactory calculus before they were faced with problems for which the classical differential geometry had not prepared them.

In the first conception of parallel displacement in a manifold, the coefficients which connect tangential directions at one point with tangential directions at another depend in a specific manner on the coefficients in the quadratic form for the square of the line element. This specific dependence characterises riemannian geometry, while the wider subject of non-riemannian geometry investigates the consequences of dependence but leaves the actual coefficients arbitrary.

In classical differential geometry the assumption that the square of the line element is positive is inevitable if the theory is supposed to be that of ordinary surfaces. In the application to the theory of relativity the fundamental quadratic form is necessarily indefinite. Prof. Eisenhart works with an indefinite form throughout, introducing a factor ± 1 explicitly when a square root has to be taken. The complication is much less serious than might be anticipated, and is completely justified by the facility with which results can be given their real interpretation.

By the preparation of these two books Prof. Eisenhart has again put us in his debt, none the less that a considerable body of mathematicians, of whom he is one of the most energetic, will be helped by them in a laudable endeavour to render them out of date as soon as possible.

Foundations of Potential Theory. By O. D. Kellogg. Pp. x, 384. RM. 19.60. 1929. (Springer)

Professor Kellogg's volume is one which will be very welcome to mathematicians. Hitherto there has been a distinct line of cleavage in the literature of potential theory. On the one hand we have the essentially practical, but not always rigorous, developments of the works on applied mathematics, and on the other hand we have the treatment of the "Cours d'Analyse", in which the physical applications are not always very illuminating. Professor Kellogg's volume bridges the gap firmly; the needs of the applied mathematician and the feelings of the pure mathematician alike receive careful handling, and to

satisfy these two requirements in under four hundred pages is no mean difficulty, but one which the author surmounts skilfully. The habit of explaining the difficulties of the problem, both mathematical and physical, before embarking on elaborate analysis is most helpful, and is indeed the key to the success of the book. Where the treatment of a problem requires methods which are not likely to be familiar to the average reader, these methods are developed briefly and lucidly, and thus one is able to pick up quite an amount of general knowledge in the course of reading the volume. For instance, there is a very nice introductory account of the theory of integral equations (in connection with the boundary value problem).

Die Theorie der Gruppen von endlicher Ordnung. By A. Speiser. Pp. viii, 194. $1.80. 1923. Die Grundlehren der mathematischen Wissenschaften, 5. (Springer)

This excellent book has now an established position. A review of the enlarged second edition appeared in vol. xiv, p. 148.

Analytische Behandeling van de Rationale Kromme van den vierden Graad in een vierdimensionale Ruimte. By J. F. De Vries. Pp. xii, 158. 4 gld. 1922. (Nijhoff)

Every curve of the second degree is a plane curve, and the simplest twisted curve is the rational cubic. Similarly, every cubic lies in a three-dimensional space, and the simplest curve to exhibit four-dimensional properties is of the fourth degree. Not only is Dr. De Vries' book in this sense the natural sequel to Wood's Cambridge Tract, but the investigations for the most part seem inevitable. For example, conical spaces of the second degree take the place of the familiar circumscribing quadric cones. This is not to say that the work could have been produced automatically. On the contrary, it reveals throughout that intimate acquaintance with four dimensions which until quite recently was almost peculiar to the inhabitants of the Flat Countries. In a concluding chapter theorems concerning the rational quartic in ordinary space are deduced by projection.

The Absolute Differential Calculus. By T. Levi-Cività. Translated by M. Long. Pp. xvi, 450. 21s. 1927. (Blackie)

Accounts of parallel displacement are now available in many languages, but the reader who prefers English to Italian will none the less be glad to have Prof. Levi-Cività's own presentation of the subject in this admirable translation, a welcome addition to the series with which Blackie and Sons are establishing a reputation as publishers and printers of mathematical books. Prof. Levi-Cività builds more slowly and carefully than many of his interpreters. The book is in three almost equal parts, devoted to introductory theories, to the calculus itself, and to physical applications; the third part was composed for this English edition.

In the first part are considered linear and other transformations, systems of differential equations, and the simplest differential quadratic form, that associated with curvilinear coordinates on a surface. Parallel displacement of a tangential vector is defined first by means of a circumscribing developable and then by means of the differential of the vector. Equations of parallelism are found, the Christoffel symbols are introduced, and the analysis is then extended to an n-dimensional manifold.

In the second part, displacement round a circuit plays a prominent part, not only in the introduction of the Riemann symbols, but in the whole theory of curvature of a manifold. Questions that are often regarded as dealing with correspondence between two manifolds are considered more logically as dealing with the assignment of different metrics to the same manifold. There are sections on manifolds of constant curvature, and a chapter on congruences of curves includes an account of some of Ricci's important investigations.

The starting point of the third part is the classical system of equations of motion of a particle in a conservative field. This system is not invariant for transformations which involve the time, and modifications, small when the velocity is small compared with that of light, lead gradually to the dynamics and statics of the general theory of relativity, and to the cosmological speculations of Einstein, de Sitter, and Schwarzschild.

Prof. Levi-Cività has been well served by Dr. Persico, on whose lecture notes the manuscript of the Italian edition was based, and by Miss Long and Dr. Dougall, the translator and proof-reader of the English edition, but it is his own clarity of thought in the first place that rendered possible so clear an exposition.

Précis d'Analyse Mathématique. I. By E. Lainé. Pp. viii, 232. 30 fr. II. By E. Lainé and G. Bouligand. Pp. viii, 336. 40 fr. 1927. (Vuibert)

An account of this course is introduced into a review of Lainé's *Exercices de Calcul*, in the *Gazette*, vol. xvi, p. 358.

Compléments et Exercices sur la Mécanique des Solides. By G. Bouligand, with the collaboration of J. Dollon. Pp. viii, 132. 18 fr. 1929. (Vuibert)

This supplement to Prof. Bouligand's *Précis de Mécanique Rationnelle*, vol. I, will be very useful to both students and teachers. There is a short section on plane kinematics, but the book is concerned principally with dynamical problems involving friction of one kind or another. Prof. Bouligand, as readers of his other text-books will expect, is at great pains to set out clearly the assumptions on which the mathematical formulation of a problem depends. The free use of vector analysis and of geometry is helpful in this respect. Cartesian coordinates may usually be wanted for the ultimate evaluation of a solution, but it is not in terms of them that ideas can be analysed. By emphasising principles, Prof. Bouligand is able to make very natural the application of Lagrange's equations to problems of non-holonomic systems.

The Story of Mathematics. By D. Larrett. Pp. 88. 3s. 6d. 1926. (Benn)

Endowed with enthusiasm, and with a sense of humour to which the quotation at the head of chap. xi bears witness, the author of this little history may infect with his own passion for mathematics readers whom scholars more learned, writers more elegant, and bibliographers with a better sense of proportion, cannot reach.

Mathematical Tables. By G. H. Bryan. Pp. 28. 3s. 6d. 1922. (Macmillan)

Most of the material was first published in the *Gazette*, May 1919.

A Treatise on Bessel Functions and their Application to Physics. By A. Gray and G. B. Mathews. Second edition prepared by A. Gray and T. M. MacRobert. Pp. xiv, 328. 36s. 1922. (Macmillan)

A student who wishes to know how Bessel functions enter into the problems of applied mathematics and physics, and to appreciate how their properties are relevant to their use, cannot do better than consult this revised edition of a well-known treatise. Apart from changes in notation, the chapters which serve this purpose so admirably are almost untouched, and a short additional chapter of the same kind describes some results of Dougall's in the theory of elasticity.

The first part of the book, in which the purely mathematical theory of the functions is developed, has been recast so completely by Dr. MacRobert that comparison with the earlier edition is almost impossible. The work is not

ambitious in scope, and it has been done as well as limitations allowed. Sets of examples have been added.

The miscellaneous examples, and the appendix giving McMahon's formulae for zeros, remain. To another appendix Stokes' investigation of asymptotic formulae has been banished, to be replaced in the text by Gibson's. A few more tables are given, the British Association's undertaking, still unfulfilled after another ten years, being the reason why further tabulation is not attempted. The addition of a few items is hardly enough to bring the bibliography up to modern standards, from which qualitatively it falls far short.

Vestiges of Pre-Metric Weights and Measures, persisting in Metric-System Europe, 1926-1927. By A. E. KENNELLY. Pp. xii, 190. 10s. 6d. 1928. (Macmillan Co.)

Armed with introductions from the Harvard Bureau of International Research, Prof. Kennelly devoted a sabbatical year to prosecuting in Europe, with great diligence, the enquiry of which this book is the outcome. He gives a formal account, complete with letters and reports in English, that is to say, usually in translation, from countless excise and other officials. There is no attempt to classify or to tabulate the results, but many of the details are interesting.

Ordinary Differential Equations. By E. L. INCE. Pp. viii, 558. 36s. 1927. (Longmans, Green)

Dr. Ince has set a new standard for English books on differential equations. Forsyth's *Theory* is of course in a class by itself, and Page's delightful little sketch is meant to be only an outline from one point of view, that of the theory of groups. The rest differ among themselves in style and rigour, but a little more attention to numerical solution here, a little less readiness to take existence theorems for granted there, hardly distinguishes one from another.

Dr. Ince's book is divided into two parts, one concerned with real variables, the other with complex. The division is not quite as sharp as it is made to seem. The elementary solutions, the use of the Wronskian and the adjoint equation, solution in series, the theory of systems, and the transformations which lead to solution by definite integrals, are all formally independent of the restriction to real variables, and the author avoids repetition simply by pointing this out. Nevertheless, the major problems are so different in the two domains that the plan adopted justifies itself.

In the real domain, existence is secured by continuity and Lipschitz conditions, and proved by successive approximations. There is an excellent brief chapter on transformation groups. The principle of solution by definite integrals is explained, and the method is applied to a number of standard cases. But it is the last two chapters of this part that specially deserve attention. The first of these is on the theory, originated by Sturm, of the distribution of zeros of real solutions of a linear equation of the form

$$\frac{d}{dx}\left\{K\frac{dy}{dx}\right\}=Gy,$$

and of the oscillation in the values of such solutions. This leads in the following chapter to the theory of characteristic values of a parameter in such an equation and of development in series of characteristic functions. We cannot be said to understand the corresponding problems in the well known particular case of Bessel functions unless we appreciate that in principle there are no analytical accidents, although it is only when a definite equation is studied that every detail becomes precise.

For the complex domain, existence theorems are established by the usual comparison with dominant functions. Singularities of individual solutions present little difficulty, but when the fundamental distinction between fixed

and movable singularities is investigated, the results are anything but concise. For the equation $w''=F(z, w, w')$, with F rational in w and w' and analytic in z, Painlevé found that if the only condition imposed is that the singularities are all to be fixed, there are fifty typical forms of the function F; in forty-four cases the equation is soluble in terms of known functions, while the other six cases introduce a new transcendental function into analysis. The attractive theory of linear equations in the complex domain is perhaps the only branch of the theory of differential equations of which every mathematician knows something; Dr. Ince deals with regular, with normal, and with subnormal integrals, and with solution by contour integration. The last three chapters are on systems of equations, on classification, and on Hille's discussion of the distribution of zeros in the plane, an extension to the complex domain of the Sturm-Liouville theory.

Naturally Dr. Ince makes no pretence to have covered the ground: "I have chosen a path which I myself have followed and found interesting". But it will be clear that the ordinary English student, who would never dream of tackling Forsyth or Schlesinger or of following up references in the *Encyklopädie*, will acquire from this book a conception of the whole subject which no other writer has attempted to give. When another edition is prepared, an amplification of the table of contents to include paragraph headings would show the scope of the volume and add to its usefulness. The exposition throughout is clear, and the printing, which was entrusted to a firm not commonly associated with this class of work, is so good that the reader is not distressed by a page which is in fact extraordinarily crowded. The difficulties caused by the author's departure to Egypt have left no trace, unless to that circumstance is due the acceptance for the Weierstrassian elliptic function of a symbol which is surely the most anaemic letter that ever failed to dominate an argument.

Vector Calculus with Applications to Physics. By J. B. Shaw. Pp. vi, 314. 14s. 1922. (Constable)

"Several systems of vector calculus have been devised, differing in their fundamental notions, their notation, and their laws of combining the symbols. . . . Disagreements arise sometimes merely with regard to . . . matters of convention; sometimes they are due to different views as to what are the important things to find expressions for; and sometimes they are due to more fundamental divergences of opinion as to the real character of the mathematical ideas underlying any system of this sort. We will indicate these differences and dispose of them in this work." The title of the book, these sentences in the opening section, and the whole of the historical sketch which forms the introductory chapter, arouse such expectation of a comparative study that when we find that the book is in fact a text-book developing one system, with details only of the notations of other systems, a readjustment of attitude is necessary before we can bring our judgment to bear.

The book is not a first course. Acquaintance with vectors is assumed, and the work begins with chapters on scalar fields and vector fields; the latter gives Poincaré's description of various singularities, without hinting at the nature of the analysis which has to be employed in their investigation. The experiment is made of introducing in a plane many of the differential ideas which are commonly explained first in a three-dimensional setting. With vectors in space we come to the quaternion. Prof. Shaw is an adherent of Hamilton. He recognises, as we all do, that the quaternions needed in practice are almost always products or quotients of vectors, but to him the functions $Sa\beta$ and $Va\beta$ are parts of the quaternion $a\beta$, not functions of the two vectors a, β. This point of view determines the sign to be attached to the scalar product, and in this way dominates formally even those parts of the book where the idea of the quaternion is not necessary. After chapters on differentiation and integration

is one on the linear vector function. This chapter opens poorly; the identity of the coefficients in the Hamiltonian cubic equation satisfied by the operator with those in the latent equation whose roots determine the invariant directions seems to be accepted as a lucky accident. But there is a careful account of the allied operators, and the chapter ends with the extension to a field operator. The reader is then equipped for the applications to classical elasticity and hydrodynamics which conclude the book.

Many writers on vectors turn so instinctively to the use of components that their work is really Cartesian analysis in disguise. Prof. Shaw thinks vectorially. That is ample recommendation of his volume.

The Number System of Arithmetic and Algebra. By D. K. Picken. Pp. viii, 76. 1923. (Melbourne Univ. Press)

It was an excellent idea to relieve the course of analysis of the burden of defining the various numbers which have to be introduced. Prof. Picken does not, however, show in this little book the logical acumen readers of the *Gazette* associate with his name. "A fractional number is defined as the quotient of two integral numbers—when such quotient is not itself integral". But (i) quotient in these circumstances is undefined, (ii) the quotient when integral cannot have the logical status of such a fraction. The two fallacies recur throughout. Again, it is claimed that "it has been found possible . . . to avoid defining one number as a class of other numbers". Why this is a laudable aim is not explained. In fact, the irrational number is left undefined; it is said to be "determined by" a Dedekind cut, and the cut is described as "a separation of all the rational numbers into two classes". The "separation" is certainly not a more tangible conception than a class. Since definitions of addition and multiplication are not given, the reader can form no conception of the technical problems involved in the development of the subject, nor is adequate foundation provided for a theory of limits.

The Story of Reckoning in the Middle Ages. By F. A. Yeldham. Pp. 96. 4s. 6d. 1926. (Harrap)

We have been advised to remember in moments of pessimism regarding the human race that an ordinary schoolboy does in a few minutes sums which only a little while ago would have occupied for days a specialist so highly trained as to be thought a magician. The processes, mechanical and mental, of the medieval computer are the subject of this little book. Part I, on the Abacus, has four chapters: (1) Notations of the Ancient East, (2) Arithmetic in Western Countries, (3) The Abacus, (4) Abacal Arithmetic with Plain Counters. In Part II, on Algorism, the chapters are: (5) The Introduction of the Hindu Notation into Western Europe, (6) The "Algorismus Vulgaris", (7) The Fundamental Rules of Algorism, (8) The Transition Years, (9) English Workers on Arithmetic before the Days of Printing.

Miss Yeldham has produced a scholarly work, delightfully written, which deserves more serious attention than it seems to have received. Perhaps the title is responsible: "Story" is always suspect.

Elementi di Aritmetica con Note storiche e numerose Questioni varie. By G. Fazzari. Sixth edition. Pp. iv, 196. L. 7. 1923. (Trimarchi, Palermo)

For a notice of an earlier edition of the first part, which deals only with whole numbers, see vol. ix, p. 264.

The complete work extends the range to fractions, vulgar and decimal.

As was said before, the exposition is as effective as it is unpretentious. The notes are scholarly and full of interest.

As to the misprints, the "fuorth" remains, but the "filth", we are glad to say, has been cleared away.

Conferencias sobre Cálculo Vectorial. By R. Gans. Pp. 60. 1926. (Univ. de La Plata)

A translation of the first two chapters and of a few later sections of Prof. Gans' well-known *Einführung*, issued by the university with which he was at one time associated. We miss the table of contents at the beginning and the table of formulae at the end, but it is a pleasure to express admiration for the printing, which is literally beautiful.

Fisica General. I. By R. G. Loyarte. Second edition. Pp. 388. $8. 1927. (Univ. de La Plata)

This, the first volume of the second edition of a five-volumed work, is a sound, uninspired text-book on mechanics, similar to most other books on the subject written from the point of view of the physicist rather than that of the mathematician. It is pleasant and easy to read, and contains excellent diagrams, descriptions of a number of interesting experiments (many of which will be unfamiliar to the mathematical reader), and no examples to be worked by the student. There are chapters on statics, dynamics, gravitation, and elasticity, a wide range being covered without much detail, and with a minimum of mathematical analysis. The reality of the subject is never lost sight of, as in the case of so many degree text-books on the subject.

Méthodes pour Résoudre les Problèmes de Géométrie. By J. Poirée. Pp. ii, 50. 1920. (Cochanaux, Auch)

The derivation of the title is classical, but the booklet is a rather incoherent collection of notes, some trivial and some stimulating. The notions of transformation and correspondence are prominent.

Cantidades Complejas. By J. I. Corral. Book I. Pp. 160. 1929. (Rambla, Bouza; Havana)

The complex number is taken in the form $r(\sin a + i \cos a)$, and de Moivre's theorem is replaced by

$$(\sin a + i \cos a)^m = i^{m-1}(\sin ma + i \cos ma).$$

Ecuaciones Numericas : Calculo de las Raices Reales. By B. Baidaff. Pp. 80. $3.50. 1926. (Martinez, Buenos Aires)

Enunciates and explains by worked examples the classical processes of discovering rational roots, segregating groups of multiple roots, enumerating real roots in a given interval, and computing irrational roots by the methods of Newton and Horner. No proofs are attempted.

BUREAU FOR THE SOLUTION OF PROBLEMS.

This is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.*, volume, page, and number. The names of those sending the questions will not be published.

CHANGE OF ADDRESS.

After December 11th, communications for the editor should be addressed to 19 Kidmore Road, Caversham, Reading.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

68

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports gifts as follows:

From Mr. T. M. A. Cooper, school books by C. V. Durell and A. W. Siddons; J. Poirée (4).

From Prof. H. G. Forder, copies of his own *School Geometry* (3 parts, 1930) and *Higher Course Geometry* (2 parts, 1931), and school books by S. Barnard and J. M. Child, T. A. Beckett and F. E. Robinson (2), W. G. Borchardt, E. Budden, G. E. St. L. Carson and D. E. Smith (2), H. H. Champion and J. A. C. Lane, R. Deakin, H. Deighton, R. C. Fawdry (2), R. C. Fawdry and C. V. Durell (2), R. W. M. Gibbs, J. McDowell, G. W. Palmer, W. Parkinson and A. J. Pressland, E. A. Price, F. Rosenberg, O. Schlömilch, A. W. Siddons and R. T. Hughes, A. N. Somerscales, A. E. Tweedy, together with:

A. F. Bentley — Linguistic Analysis of Mathematics - - - 1932

C. J. Keyser — Human Worth of Rigorous Thinking - - - 1916

H. Minkowski — Grundgleichungen der Elektrodynamik - - - 1910

J. Nicod — Foundations of Geometry and Induction - - - 1930

Translated from French by P. P. Wiener, to form a volume in the *International Library of Psychology, Philosophy and Scientific Method.*

H. v. Sanden — Darstellende Geometrie - Teubners Leitfäden 2 1931

J. B. Shaw — Philosophy of Mathematics - - - - 1918

E. Toulouse — Henri Poincaré - - - - - - - [1910]

"Enquête médico-psychologique sur la supériorité intellectuelle."

H. Weyl — Raum Zeit Materie (2, *i.e.* 1 (1918) rep.) - 1919

From Dr. H. B. Heywood:

H. T. Davis — Volterra Integral Equation of Second Kind - - 1930

W. V. Lovitt — Linear Integral Equations - - - - - 1924

V. Volterra et J. Pérès.

Composition et Fonctions Permutables - - Borel 1924

From Prof. E. H. Neville:

A. S. Eddington — Expanding Universe - - - - - - 1933

From Dr. S. A. Peyton:

H. Breen — Practical Astronomy - - - - - - - 1856

J. R. Young — Algebra - - - - - - - - 1834

From Prof. G. N. Watson, to commemorate his term of office as President:

G. C. de' T. di Fagnano

Opere Matematiche (3 vols.) - - 1911, 1911, 1912

The retiring President has made his gift complete by having the volumes durably bound.

The following has been bought:

P. Tannery — Mémoires Scientifiques: III - - - - 1915

The *Mémoires* of Paul Tannery, edited by two of the other three great scholars of Greek mathematics, Heiberg and Zeuthen—the third is of course Heath—occupy eleven volumes, but the first three of these form a collection complete in itself, under the heading *Sciences Exactes dans l'Antiquité*; since the first two volumes came to the Association from Mr. Greenstreet, to pick up the third as an odd volume was a stroke of luck.

LONDON BRANCH.

The opening meeting of the autumn session was held on Oct. 7th, 1933, at Bedford College, when Mr. W. C. Fletcher spoke on " How can we introduce solid geometry into our teaching ? " There were present some 70 members and visitors.

Mr. Fletcher suggested that the customary school course in geometry began at the wrong end. It should begin with concrete experience of three-dimensional things. To begin with a long course of plane geometry was apt to produce an actual blindness to the reading of three-dimensional diagrams. The power of making such diagrams was one that had to be acquired ; it did not come naturally. Simple practice could be given by sketching a room and objects related to it. Exercises on shadows might follow. These would bring out the fact that a point in a three-dimensional diagram meant nothing without some further specification as to its position. Many interesting results giving properties of plane figures—for example, Desargues' theorem and Pappus' theorem—followed from solid figures. The properties of transversals and of parallels appeared more significantly from such work than from the customary elementary exercises in plane geometry. The generalisations of the conventional solid geometry course, *e.g.* the notion and definition of a plane, could be attained through a course of such work. Naturally some teaching of " plan and elevation " would come in. Mr. Fletcher concluded with a plea for experiment in this wide and neglected field.

On Nov. 11th, Mrs. E. M. Williams, of the Education Department of King's College, spoke on " The geometrical notions of young children ". She illustrated some experiments on the first notions of symmetry, in which young children fitted various geometrical shapes into formboards and into a wheel which could be rotated into different positions. It appeared that symmetry was most easily realized when the axis of symmetry was vertical. It was interesting to compare the methods used by the children. The cooperation of teachers in following up this and similar researches was invited.

C. T. Daltry (Hon. Sec.).

MIDLAND BRANCH.

During the year 1932-1933, the following meetings were held :

1932.

July 2nd. A social meeting at Solihull School and a discussion on the Additional Mathematics syllabus of the Northern Universities Joint Board.

Oct. 19th. The President, Mr. Pratt, gave an address on " The human touch in the teaching of mathematics ".

Nov. 25th. Mr. Peek on " Lunar Occultations ".

1933.

Feb. 8th. Miss McDiarmid on " Dimensions ".

March 24th. The Annual General Meeting, election of officers and a paper by Mr. Kuttner on " The five-colour map theorem ".

The average attendance has been about 25, and the number of members and of associates are each about 30. Any one in the Midland area wishing to join the branch should communicate with one of the Secretaries.

During this session the following meetings have been held :

1933.

Oct. 18th. A discussion on the mathematics section in the report of the Investigators into the School Certificate examinations, opened by Mr. Fulford.

Nov. 22nd. Mr. Bates on " Mr. Dudeney at the Breakfast Table ".

The business meeting will be held on Jan. 31st, 1934, and Mr. Preece will speak on "Examples of Euler's Work". On March 14th, Dr. Askwith will speak on "Geometry". A. Jackson (Hon. Sec.).

SOUTHAMPTON BRANCH.

The number of members of the branch is 2, and the number of associates 25.

On November 24th, 1932, Brigadier Winterbotham, R.E., Director of the Ordnance Survey, addressed the branch (and guests from the Geographical and Engineering Societies) on "The Geodesy of the Empire". On January 7th, 1933, Professor Howland, President of the branch, led a discussion on "The suitability of the London External Syllabus for the General Degree in respect of students training to be teachers of mathematics in schools". On February 17th, 1933, Mr. Pars, of Jesus College, Cambridge, read a paper on the "Four Colour Problem". On March 7th, 1933, Professor Levy, of the Imperial College of Science, read a paper on "An empirical approach to mathematics", which led to much discussion. On June 23rd, 1933, Mr. Worrall, of the Municipal College, Portsmouth, read a paper on "Continuity in curves", dealing with practical graphs of curves of the type $y=x^{1/n}$.

Randal Casson (Hon. Sec.).

YORKSHIRE BRANCH.

The Branch held its Autumn (Annual) Meeting on Saturday, November 18th, 1933. Mr. Gilham was in the Chair; there were 53 members present.

The minutes were read and signed. Ten new members were elected. The Treasurer submitted the statement of accounts and these were passed. The Treasurer, who was retiring, and the Auditor were thanked for their splendid work.

The following officers were elected for the year:

President: Dr. R. Stoneley, Leeds University.
Vice-Presidents: Mr. W. Todd, Mirfield Grammar School.
Mrs. J. L. Brown, Leeds Training College.
Hon. Treasurer: Miss N. G. Shapley, Leeds Girls' High School.
Hon. Secretary: Mr. J. D. Edington, York Training College.

Committee: Mr. Gabriel, Mr. Barnes, Mr. Cooper, Mr. Montagnon, Miss Bowman, Miss Mathews, Mr. Lamb, Col. Mozley, Mr. Wilkinson.

Mr. C. W. Gilham then read a paper, entitled "Abel's Life and Times", which was very much appreciated.

A committee was formed to discuss the question of Trigonometry in Schools and to report to the Meeting their findings—this to form a subject of discussion at a subsequent meeting. J. D. Edington (Hon. Sec.).

CORRIGENDUM.

In Part I of the biographical account of E. M. Langley which appeared in the number for October, 1933, for "triangle" read "tringle", a word used by E. M. L. to indicate a makeshift.

THE NEW ERA.

The January number of *The New Era* is a 'special mathematics number' edited for the occasion by Dr. P. B. Ballard. Among the articles are two from members of the Mathematical Association: Mr. A. L. Atkin on "The Teaching of Elementary Geometry" and Mr. C. V. Durell on "Algebra and General Education." The number is an excellent sixpennyworth.

JOURNALS RECEIVED.

When no number is attached, no part has been received since a previous acknowledgment.

Abhandlungen aus dem Math. Sem. der Hamburgischen Universität.
American Journal of Mathematics. 55 : 4.
American Mathematical Monthly. 40 : 7, 8, 9.
Anales de la Sociedad Cientifica Argentina. 116 : 3, 4, 5.
Annales de la Société Polonaise de Mathématique.
Annali della R. Scuola di Pisa. Ser. 2. 2 : 4.
Annals of Mathematics. Ser. 2. 34 : 4.
Berichte über die Verhandlungen der Akad. der Wiss. zu Leipzig : Math. Phys. Klasse.
Boletin Matematico. 6 : 2, 3, 4, 5, 6, 7.
Boletin Matematico Elemental. 4 : 6, 7, 8, 9.
Boletin del Seminario Matemático Argentino.
Bollettino della Unione Matematica Italiana. 12 : 4, 5.
Bulletin of the American Mathematical Society. 39 : 9, 10, 11.
Bulletin of the Calcutta Mathematical Society. 24 : 4 ; 25 : 1.
Communications . . . de Kharkow et . . . de l'Ukraine. Ser. 4. 6.
Contribución al Estudio de las Ciencias Fisicas y Matemáticas.
L'Enseignement Mathématique.
Ergebnisse eines Mathematischen Kolloquiums (Wien). 5.
Gazeta Matematica. 39 : 1, 2, 3, 4.
Half-Yearly Journal of the Mysore University.
Jahresbericht der Deutschen Mathematiker-Vereinigung.
Japanese Journal of Mathematics. 9 : 3, 4.
Journal of the Indian Mathematical Society.
Journal of the London Mathematical Society. 8 : 4.
Journal of the Mathematical Association of Japan. 15 : 4-5.
Mathematical Notes. 28.
Mathematics Student. 1 : 2, 3.
Mathematics Teacher. 26 : 6, 7, 8.
Memoria (Univ. Nac. de la Plata).
Monatshefte für Mathematik und Physik.
Nieuw Archief voor Wiskunde. 18 : 1.
Periodico di Matematiche.
Proceedings of the Edinburgh Mathematical Society. Ser. 2. 3 : 4.
Proceedings of the Physico-Mathematical Society of Japan. Ser. 3. 15 : 8, 9, 10, 11, 12.
Publicaciones . . . Físico-Matemáticas . . . de la Plata.
Publications de la Faculté des Sciences de Masaryk. 174, 176, 178, 181, 184.
Revista de Ciencias (Peru).
Revista Matemática Hispano-Americana (Madrid). Ser. 2. 8 : 5-6.
Revue Semestrielle des Publications Mathématiques. 36 : 2 ; 38 : 4, 5.
School Science and Mathematics. 33 : 7, 8, 9.
Scripta Mathematica. 2 : 1.
Sitzungsberichte der Berliner Mathematischen Gesellschaft. 32.
Studia Mathematica.
Unterrichtsblätter für Mathematik und Naturwissenschaften. 39 : 8, 9, 10.
Wiskundige Opgaven met de Oplossingen.

BOOKS RECEIVED FOR REVIEW.

H. F. Baker. *Principles of geometry. VI. Introduction to the theory of algebraic surfaces and higher loci.* Pp. ix, 308. 17s. 6d. 1933. (Cambridge)

P. B. Ballard and J. Brown. *The London Arithmetics. First series. Pupil's book. I, II, III.* Pp. 64 each. Paper 8d., cloth 10d. each. *Pupil's book. IV.* Pp. 80. Paper 10d., cloth 1s. *Teacher's book. I.* Pp. 79. 2s. *Teacher's book. II.* Pp. 72. 2s. *Teacher's book. III.* Pp. 76. 2s. *Teacher's book. IV.* Pp. 90. 2s. 3d. Answers separately, 9d. each part. 1934. (University of London Press)

K. Bartel. *Malerische Perspektive. I.* Pp. viii, 339. Geb. RM. 16. 1934. (Teubner)

G. Bauer und L. Bieberbach. *Vorlesungen über Algebra.* 5th edition. Pp. x, 358. RM. 14. 1933. (Teubner)

H. Behnke und P. Thullen. *Theorie der Funktionen mehrerer komplexer Veränderlichen.* Pp. vii, 115. RM. 13.80. 1934. Ergebnisse der Mathematik, Band III, Heft 3. (Springer)

V. Bernstein. *Leçons sur les progrès récents de la théorie des séries de Dirichlet.* Pp. xv, 320. 60 fr. 1933. (Gauthier-Villars)

H. F. Biggs. *The electromagnetic field.* Pp. viii, 158. 10s. 6d. 1934. (Oxford)

M. Black. *The nature of mathematics.* Pp. xiv, 219. 10s. 6d. 1933. (Kegan Paul)

G. A. Bliss. *Algebraic functions.* Pp. ix, 218. $3.00. 1933. American Math. Soc. Colloquium Publications, 16. (American Math. Society)

O. Bolza. *Vorlesungen über Variationsrechnung.* Pp. ix, 705, 13. Geb. RM. 20. Reprint of 1909 edition. 1933. (Koehlers Antiquarium, Leipzig)

T. Bonnesen und W. Fenchel. *Theorie der konvexen Körper.* Pp. vii, 164. RM. 18.80. 1934. Ergebnisse der Mathematik, Band III, Heft 1. (Springer)

F. Bücking. *Das bizentrische Viereck.* Pp. iv, 44 and 6 plates. RM. 3.60. 1933. (Teubner)

P. Copel. *Éléments d'Optique géométrique.* Pp. ix, 206. 25 fr. 1933. (Gauthier-Villars)

C. J. Cozens. *Mathematical test papers.* (*School Certificate Standard.*) 1s. *Answers.* Pp. 16. 6d. 1934. (Arnold)

L. Crosland. *Revision mathematics.* Pp. viii, 254. With answers, 3s. 6d. 1934. (Macmillan)

H. T. Davis. *Tables of the higher mathematical functions. I.* Pp. xiii, 377. 25s. 1933. (Williams and Norgate; Principia Press, Bloomington, Indiana)

C. V. Durell and A. Robson. *Elementary calculus. II.* Pp. xii, 241-548. 7s. 6d. Without appendix, 6s. 6d. 1934. (Bell)

L. R. Ford. *Differential equations.* Pp. x, 263. 15s. 1933. (McGraw-Hill)

J. Frenkel. *Wave mechanics. Advanced general theory.* Pp. viii, 525. 35s. 1934. International series of monographs on physics. (Oxford)

R. J. Fulford. *Revision and mental tests in geometry.* Pp. viii, 73. 1s. 1933. (University Tutorial Press)

E. Jahnke und F. Emde. *Funktionentafeln mit Formeln und Kurven.* 2nd edition, revised. Pp. xviii, 330. Geb. RM. 16. 1933. (Teubner)

F. W. Johnson. *Easily interpolated trigonometric tables with non-interpolating logs, cologs and antilogs.* Paper $1.35. Fabrikoid $3.50. 1933. (Simplified Series Publishing Co.)

C. Kuratowski. *Topologie. I. Espaces métrisables, espaces complets.* Pp. x, 285. $4.50. 1933. Monografje Matematyczne, 3. (Warsaw)

J. Lense. *Reihenentwicklungen in der mathematischen Physik.* Pp. 178. RM. 9.50. 1933. (Walter de Gruyter, Berlin)

E. Mallett. *Vectors or electrical engineers.* Pp. viii, 181. 13s. 6d. 1933. (Chapman and Hall)

A. B. Mayne. *The essentials of school geometry.* Pp. xiv, 408, ix. 4s. 6d. 1933. (Macmillan)

L. M. Milne-Thomson. *The calculus of finite differences.* Pp. xix, 558. 30s. 1933. (Macmillan)

Sir I. Newton. *Mathematische Principien der Naturlehre.* Pp. viii, 666. Geb. RM. 20. Reprint of 1872 translation. 1933. (Koehlers Antiquarium, Leipzig)

H. Phillips, S. T. Shovelton and C. S. Marshall. *Caliban's Problem Book.* Pp. x, 330. 6s. 1933. (De La Rue)

A. S. Ramsey. *Statics.* Pp. xi, 296. 10s. 6d. 1934. (Cambridge)

J. Ser. *Les calculs formels des séries de factorielles.* Pp. vii, 98. 20 fr. 1933. (Gauthier-Villars)

F. Severi. *Lezioni di Analisi.* Pp. viii, 434. L. 75. 1933. (Zanichelli, Bologna)

A. W. Siddons and C. T. Daltry. *Elementary algebra. II.* Pp. viii, 133-372, xxiii-lxxiii. 3s. 6d. Without answers, 3s. 1934. (Cambridge)

W. Sierpiński. *Introduction to general topology.* Pp. x, 238. 17s. 1934. (University of Toronto Press; Oxford University Press)

D. M. Y. Sommerville. *Analytical geometry of three dimensions.* Pp. xvi, 416. 18s. 1934. (Cambridge)

D. J. Struik. *Theory of linear connections.* Pp. vii, 68. RM. 8.60. 1934. Ergebnisse der Mathematik, Band III, Heft 2. (Springer)

M. Walker. *Conjugate functions for engineers.* Pp. 116. 12s. 6d. 1933. (Oxford)

LONDON BRANCH.

THE Presidential Address on "The Finite and the Infinite" was given by Professor S. Brodetsky (Professor of Applied Mathematics in the University of Leeds) at Bedford College on 9th December. Mr. Boon was in the chair, and about 50 members and visitors were present. The speaker began by tracing the development of the idea of an infinite universe. About 1917 this gave place to a conception of a finite and bounded universe. Thus a primitive instinctive belief in infinity was superseded by a highly complex conception. Indeed, there seemed to be a sort of Boyle's Law between scientific knowledge and belief in infinity. This could be illustrated by the history of the velocity of light, and of the infinite divisibility of matter. From instinctive belief in an infinite velocity we had attained, through discovering that the velocity was finite, the results of the theory of relativity. From instinctive belief in the infinite divisibility of matter we had passed through knowledge of the relation between a particle of matter and a train of waves to the complexities of quantum theory and statistical mechanics. Thus the infinitely small had been eliminated and replaced by the finite. On the other hand the mathematical theory of the infinite developed by Cantor appeared to lead to final concepts of infinity. Only through pure mathematics was it possible to attain right lines of progress for the study of the infinite.

An informal discussion followed Professor Brodetsky's brilliant address.

Annual Business and Members' Topics were considered on 3rd February. The Committee's Report showed that membership had fluctuated around 180 members and 110 associates. The Treasurer's Report showed a healthy balance of some £20. It was announced that Arthur R. Hinks, Esq., Gresham Professor of Astronomy, had accepted the office of President for 1934. Miss Yeldham had been compelled to retire from the secretaryship owing to illness, after many years of devoted and painstaking service. The remaining officers were re-elected. As Members' Topics, Mr. Atkinson showed examples of slide rules constructed from graphs, Mr. Snell outlined some neat applications of simple vector analysis, Mr. Styler illustrated how essays could be used for revision and for summarising work in mathematics, and Miss King discussed the use and abuse of cancelling.

At the meeting on 24th February at Bedford College a rather small audience of 32 heard a lively and stimulating paper from Mr. H. E. Piggott of the Royal Naval College, Dartmouth, dealing with "Some Ideas on Momentum and Energy". The first part of the paper dealt with the history of these concepts, and pointed out that for more than half a century there was vigorous controversy as to their nature. The second part discussed teaching hints, illustrated by reference to recent articles in the *Gazette*, and to the writings of Sir George Greenhill. An interesting problem was presented by the times taken by a train to run on, and off, an incline. Finally, two more elaborate problems were analysed in detail, cyclostyled copies of the mathematics being distributed. The first considered the energy acquired by a shot fired from a moving ship, and revealed several unexpected consequences. The second was the experiment on impact referred to in the *Report on the Teaching of Mechanics*, p. 67.

C. T. Daltry (Hon. Sec.).

QUEENSLAND BRANCH.

Report for 1932-1933, presented to the Annual Meeting, 31st March, 1933.

I have the honour to present to you the eleventh Annual Report of the Queensland Branch of the Mathematical Association.

The Annual Meeting was held on 8th April, 1932, at the University. The Annual Report and the financial statement were presented and adopted, and officers for the ensuing year were elected. Dr. E. F. Simonds addressed the meeting on "Mathematical Statistics".

During the year three ordinary meetings were held. The first was held at the Boys' Grammar School, Gregory Terrace, and at this meeting Mr. A. J. Stoney read a paper on "Mathematics in Surveying". At the second, held also at the Boys' Grammar School on 5th August, Mr. J. P. McCarthy read a paper on "Pythagorean Arithmetic", and at the third, held at the University on 28th October, Mr. J. C. Deeney spoke on "Primitive Time Reckoning".

The number of members is 29, of whom 8 are members of the Mathematical Association. Copies of the *Mathematical Gazette* come to hand from time to time and are circulated among associates. The statement of receipts and expenses shows a credit balance of £4 2s. 10d., which is about £2 9s. less than at the end of our last year. High exchange on subscriptions to London and a falling off in local subscriptions account for the drop. Attendance at meetings has been maintained, and I desire to thank those members who have been so good as to give their time to the preparation of papers for the various meetings.

J. P. McCarthy (Hon. Sec.).

COMPOSITIO MATHEMATICA.

The international character of the editorial board of this new journal should guarantee a high standard. The services of forty-seven distinguished mathematicians have been secured, the representatives of this country being G. H. Hardy, E. T. Whittaker and B. M. Wilson. The members of the administrative committee are L. Bieberbach, L. E. J. Brouwer, Th. De Donder, G. Julia and B. M. Wilson; the publisher is Noordhoff of Groningen, and the price, per volume of 480 pages, is 20 Dutch guilders.

The first part of Volume I has just appeared and contains in its 200 pages papers by P. Lévy, van der Corput, Watson, Wavre, Doetsch, Hille and Tamarkin, Fubini, von Neumann, Levi-Civita, Bourion, Khintchine, Bosanquet and Offord, and Loewy. If the promise of this first part can be maintained, *Compositio Mathematica* should soon rank high among the mathematical periodicals of the world.

CORRESPONDENCE.

28 Storey's Way,
Cambridge, *January* 27, 1934.

To the Editor of the *Mathematical Gazette*.

Dear Sir,

I am not sorry that the report which was taken of what I said at the meeting of the Mathematical Association in London on January 5th is too long for publication in the *Gazette*, for, as the result of conversations and correspondence since the meeting, I have come to the conclusion that for *elementary* teaching the traditional method has great advantages, combining so happily the practical with the theoretical.

So far as I have been able to glean opinion it is this, that for beginners the old method of starting with a ready-made plane surface is the best, and the purely theoretical method as outlined in my paper should come later, and should be for those who are able to profit by it.

I am glad to have got this clear in my mind. For I want, indeed we all want, to do the thing that is best in the interests of education.—Yours truly,

E. H. Askwith.

BUREAU FOR THE SOLUTION OF PROBLEMS.

This is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.*, volume, page, and number. The names of those sending the questions will not be published.

THE LIBRARY.

Donations of school books, old or new, are always welcome.

GLEANINGS: AN APPEAL.

The Editor will be grateful for help in the filling up of odd corners. A precise reference should accompany every quotation.

BOOKS RECEIVED FOR REVIEW

N. Bohr. *Atomic theory and the description of nature.* Pp. 119. 6s. 1934. (Cambridge)

N. B. Conkwright. *Differential equations.* Pp. xii, 234. 7s. 6d. 1934. (Macmillan)

V. Cornish. *Ocean waves and kindred geophysical phenomena.* Pp. xv, 164. 10s. 1934. (Cambridge)

J. G. Crowther. *The progress of science.* Pp. x, 304. 12s. 6d. 1934. (Kegan Paul)

N. R. C. Dockeray. *An elementary treatise on pure mathematics.* Pp. xiv, 566 16s. 1934. (Bell)

C. V. Durell and A. Robson. *Higher Certificate calculus.* Pp. xi, 241-368, xvi. 4s. Part I separately, 1s. 6d. 1934. (Bell)

R. C. Fawdry. *Examples in elementary statics and dynamics.* Pp. vii, 145, xviii. 3s. 6d. 1934. (Bell)

N. M. Gunther. *La Théorie du Potentiel et ses applications aux problèmes de la physique mathématique.* Pp. 303. 70 fr. 1934. (Gauthier-Villars)

D. Hilbert und P. Bernays. *Grundlagen der Mathematik. I.* Pp. xii, 471. RM. 36; geb. RM. 37.80. 1934. Grundlehren der math. Wiss., 40. (Springer)

A. Hurwitz. *Mathematische Werke. I. Funktionentheorie.* Pp. xxiv, 734. 1932. *II. Zahlentheorie, Algebra und Geometrie.* Pp. xiv, 755. 1933. Schwfr. 80; geb. schwfr. 88. (Birkhäuser, Basel)

T. E. Jessop. *A bibliography of George Berkeley.* With an inventory of Berkeley's manuscript remains by A. A. Luce. Pp. xvi, 99. 7s. 6d. 1934. (Oxford)

L. Koschmieder. *Variationsrechnung. I.* Pp. 128. RM. 1.62. 1933. Sammlung Göschen, 1074. (Walter de Gruyter, Berlin)

E. Landau. *Grundlagen der Analysis.* Pp. xiv, 134. Geh. RM. 7.70; geb. RM. 8.80. 1930. (Akademische Verlagsgesellschaft, Leipzig)

E. Landau. *Einführung in die Differentialrechnung und Integralrechnung.* Pp. 368. Geh. RM. 20; geb. RM. 22.50. 1934. (Noordhoff, Groningen)

D. Larrett. *School Certificate algebra.* With answers, 5s.; without answers, 4s. 6d. 1934. (Harrap)

H. Liebmann. *Synthetische Geometrie.* Pp. viii, 119. RM. 5.60. 1934. Teubners mathematische Leitfäden, 40. (Teubner)

H. J. Mann and J. S. Norman. *Algebra.* New edition. Pp. viii, 279. 4s.; with answers, 4s. 6d. 1934. (Deane)

G. Prasad. *Some great mathematicians of the nineteenth century. II.* Pp. xviii, 324. RM. 6. 1934. (Benares Mathematical Society; Koehlers Antiquarium, Leipzig)

F. A. J. Rivett. *A new arithmetic.* Pp. 343, 30. With answers, 5s.; without answers, 4s. 6d. 1934. Or in two parts: *A new junior arithmetic.* With answers, 2s. 6d.; without answers, 2s. (1932.) *A new senior arithmetic.* With answers, 3s.; without answers, 2s. 6d. (Arnold)

H. Seifert und W. Threlfall. *Lehrbuch der Topologie.* Pp. vii, 353. Geb. RM. 20. 1934. (Teubner)

J. Shibli. *Recent developments in the teaching of geometry.* Pp. viii, 252. $2.25. 1932. (State College, Pennsylvania)

W. Sierpiński. *Hypothèse du Continu.* Pp. v, 192. $3.50. 1934. Monografje Matematyczne, 4. (Warsaw)

I. S. Sokolnikoff and E. S. Sokolnikoff. *Higher mathematics for engineers and physicists.* Pp. xiii, 482. 24s. 1934. (McGraw-Hill)

E. Steinitz und H. Rademacher. *Vorlesungen über die Theorie der Polyeder.* Pp. viii, 351. Geh. RM. 27; geb. RM. 28.80. 1934. Grundlehren der math. Wiss., 41. (Springer)

H. J. Tappenden. *Reversions and life interests.* Pp. xii, 57. 7s. 6d. 1934. (Cambridge)

MATHEMATICS AT THE BRITISH ASSOCIATION, 1934.

EXCEPT for papers by Prof. J. A. Carroll and Dr. W. L. Marr, the attraction of the Aberdeen programme for the mathematician is on the side of mathematical physics. Three discussions are being arranged :

(1) The electronic theory of metals.—Prof. C. G. Darwin, Prof. R. H. Fowler, Prof. N. F. Mott ;

(2) Unified field theories in physics.—Prof. M. Born, Mr. J. H. C. Whitehead, Prof. E. T. Whittaker ;

(3) The fine-structure constant and the ratio of the masses of the electron and the proton.—Dr. H. Bethe, Prof. C. G. Darwin, Sir Arthur Eddington, Prof. E. T. Whittaker.

A programme which attempts to provide a little light refreshment for everybody satifies nobody ; this year the organisers offer a rich feast to mathematicians interested in theories of matter. It is to be remembered that Sir James Jeans is President of the Association, and Prof. H. M. Macdonald of Section A.

SYDNEY BRANCH.

REPORT FOR THE YEAR 1933.

THE membership of the Sydney Branch now stands as follows :

Members of the parent Association, 16 ; associates, 101.

During the year two meetings were held. At the first of these, Professor Wellish spoke on "Some Aspects of the Theory of Relativity". In his address, Professor Wellish ably traced the development of the concepts of mechanics from the pre-Newtonian stage through Newton and then on to the development of the Tensor calculus. At the same meeting, Mr. R. J. Lyons was appointed as the Sydney representative to the Branches Committee.

At the Annual Meeting held in December, reports were received from the Hon. Treasurer and the joint Hon. Secretaries. Office-bearers for 1934 were appointed as follows : *President* : Professor H. S. Carslaw ; *Hon. Treasurer* : Mr. A. L. Nairn ; *Hon. Secretaries* : Miss E. A. West, Mr. H. J. Meldrum.

Dr. R. L. Aston gave a very interesting address on "Tacheometry". His exposition of some of the newer as well as some of the older methods used by surveyors was made very clear, with the help of diagrams. Some modern instruments were present on exhibition.

The distribution of the *Gazette* has proceeded satisfactorily as an important part of the Branch's activities.

E. A. WEST,
H. J. MELDRUM } Hon. Secs.

VICTORIA BRANCH.

REPORT FOR THE YEAR 1933.

FOUR ordinary meetings were held in the year, at which there were good attendances.

On 25th April, Professor Kernot gave a paper on "Mechanical Paradoxes". He showed diagrams of perpetual motion machines, and many peculiar models, such as a cylinder which rolled uphill, and he also illustrated ordinary physical theory such as cohesion.

On 13th June, Professor Cherry gave a paper on "The Snub Cube", including a general talk about the archimidean semi-regular solids and showing cardboard models of some of them. The snub cube is a polyhedron enclosed by 12 equal squares, 8 equal hexagons, and 6 equal octagons. At each solid angle a square, hexagon and octagon meet.

On 17th July, Mr. Picken gave short papers on "Products of Vectors" and on "Forsyth's extended Leibnitz theorem".

On 18th September, Dr. Baldwin gave a most interesting account of majority systems of election using preferential voting. Dr. Baldwin explained various methods by which the votes were counted and pointed out how easily

most of them might lead to unsatisfactory results. He described in particular practical methods of counting the votes on the Nanson system in which a count is made of the number of times a preference is recorded for any candidate A over any other candidate N. He showed how the successive scrutinies rejecting or electing successive candidates could be rapidly carried out. The difficulties arose, not in the case of a large body of electors, but when there was a large number of candidates. For example he spoke of some elections in which there were 30 or 40 candidates. In such a case the successive scrutinies became troublesome.

The Association was invited by Miss Laing to hold its Annual Meeting at her house at Macedon and a very enjoyable excursion resulted.

Office-bearers for 1934 were elected as follows: *Hon. President*: Professor Nanson; *President*: Mr. R. J. A. Barnard; *Vice-Presidents*: Professor J. H. Mitchell, Professor T. M. Cherry, Dr. J. M. Baldwin, Mr. D. K. Picken, Miss A. Laing; *Committee*: Miss J. T. Flynn, Miss W. Waddell, Mr. H. M. Campbell, Mr. Trudinger; *Secretary*: Mr. J. A. Seitz; *Treasurer and Assistant Secretary*: Mr. Palfreyman.

R. J. A. Barnard, Hon. Sec.

YORKSHIRE BRANCH.

The Branch held its Spring Meeting on Saturday, 3rd February, 1934, at The University, Leeds. Dr. Stoneley was in the chair and there were forty-six members present. The meeting was devoted to the discussion on a report of a sub-committee on Trigonometry in Northern Universities' School Certificate; Miss K. Reeve and Mr. Montagnon opened the discussion. Finally the following resolution was passed:

"That certain specialised topics, such as Stocks and Shares, Compound Interest, etc., be no longer compulsory in the Group III Mathematics (Arithmetic) paper of the School Certificate of the Joint Matriculation Board of the Universities of Manchester, Liverpool, Leeds, Sheffield and Birmingham, and that a sufficient number of alternative questions on Trigonometry (including both numerical and theoretical trigonometry) be set in section B to enable a candidate to omit consideration of these specialised topics if he so wish. It is desirable that the questions set in section B should be of a less complicated type than customarily set".

The Branch held its Annual Dinner on 17th March, 1934, at the Griffin Hotel, Leeds. Dr. Stoneley presided and there were thirty-seven members and friends present. Professor and Mrs. Roberts of The Royal Military Academy, Woolwich, were the Guests of the Branch. Lt.-Col. E. N. Mozley, D.S.O., proposed the toast "Our Guests", Professor Edwards supported this and Professor Roberts replied. Professor Brodetsky proposed the toast of "Miss Applied Mathematics" and Miss Bowman replied. Mr. Montagnon proposed the toast of "The President" and Dr. Stoneley replied. Professor Milne then spoke on the forthcoming departure of Dr. Stoneley, paying tribute to the work of Dr. Stoneley, while at Leeds University and emphasizing that Leeds' loss was Cambridge's gain. Songs were given by Mr. Hardy and Mr. Montagnon acted as toastmaster.

The Branch held its Summer Meeting on Saturday, 12th May, 1934, at Thornes House Secondary School, Wakefield. Dr. Stoneley was in the chair and there were twenty-three members present. Mr. Montagnon was elected president, Dr. Stoneley being unable to complete his term of office, owing to his appointment at Cambridge. Dr. Stoneley gave a particularly fine paper on "The Interior of the Earth"; Professor Brodetsky proposed a vote of thanks to Dr. Stoneley and this was supported by Lt.-Col. Mozley and Mr. Liddle. Dr. Stoneley moved and Lt.-Col. Mozley supported a vote of thanks to Mr. Liddle, staff and governors for their hospitality. The Branch was entertained to tea and then inspected the fine gardens surrounding the School.

J. D. Edington, Hon. Sec.

THE FIFTEENTH ANNUAL MEETING OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, CLEVELAND, OHIO, 23RD AND 24TH FEBRUARY, 1934.

THE following papers were presented at the Fifteenth Annual Meeting: "The Future of Geometry in the High School", by Ralph Beatley; "The Future of Geometry in the High School", by Roland R. Smith; "A 'Panel' Discussion on the Present Crisis in Secondary Mathematics", by 10 members of the Board of Directors; "The Problems of Ability Grouping, Administrative Phases", by C. M. Stokes; "Remedial Work in Arithmetic", by Genevieve Skehan; "Problem of Individual Differences", by Clara E. Murphy; "An Experiment in Teaching Graphs", by Mrs. W. E. Pitcher; "What Can We Do to Meet the Challenge of the Present Situation in Secondary Mathematics", by William D. Reeve; "A Report of the Policy Committee", by J. O. Hassler; "Mathematics and Music", by Professor Carl A. Garabedian.

The results of the annual election: *President*: J. O. Hassler, Professor of Mathematics, University of Oklahoma; *Second Vice-President*: Allen R. Congdon, University of Nebraska; *Members of the Board of Directors*: William Betz, Rochester, New York, H. C. Christofferson, Oxford, Ohio; Edith Woolsey, Minneapolis, Minnesota; Martha Hildebrandt, Maywood, Illinois.

EDWIN W. SCHREIBER, Secretary.

VOLUNTEERS WANTED.

THE British Association Committee on Mathematical Tables has in the press a table giving the complete decomposition into prime factors of every number less than 100,000. The cost of production is being defrayed from a bequest of the late Lt.-Col. A. J. C. Cunningham, and it is proposed to publish the volumes at less than cost price in order that some of the benefit of the bequest may be passed on to the user of the tables.

The problem of securing accuracy in a factor table is altogether different from the problem for an ordinary functional table. An error in a functional table is shown up when the table is differenced, without reference to the origin of the entries; moreover, a substantial isolated error is usually obvious to anyone who has occasion to employ the numerical values, and is unlikely to do harm beyond humiliating the producer. In a factor table, no check is provided by a comparison of adjacent entries, nor is the user to suspect at a glance that 37 is a misprint for 47. In short, there is no substitute for perfect accuracy in calculating and printing.

For the volume in the press, the computation has been performed independently in triplicate; the chance that three persons not in touch with each other should make the same mistake is surely negligible, and if the printed sheets are read against each of the three manuscripts, the possibility of error will be eliminated if only the comparisons are reliable, but it is necessary to have at least two readings in each case to give confidence. This is a colossal task, and if it is attempted by a small group of people it will be a very long one, since it is all but impossible to maintain close attention if long spells are attempted, however anxious one may be to get on with the work.

The calculations having been voluntary in the first place, the Committee is loath to engage professional assistance at this stage, if, as it seems reasonable to suppose, there are a great many people interested in numbers who would gladly take a small share in producing a table of permanent interest and value. Will any reader who will undertake the responsibility for even a few pages write direct to the Secretary of the Committee, Dr. L. J. Comrie, Nautical Almanac Office, S.E. 10.

JOURNALS RECEIVED.

When no number is attached, no part has been received since a previous acknowledgment.

Abhandlungen aus dem Math. Sem. der Hamburgischen Universität.
American Journal of Mathematics. 56 : 1, 2, 3.
American Mathematical Monthly. 40 : 10, i; 41 : 1, 2, 3, 4, 5, 6.
Anales de la Sociedad Cientifica Argentina. 116 : 6; 117 : 1, 2.
Annales de la Société Polonaise de Mathématique. 8; 10; 11.
Annali della R. Scuola di Pisa. Ser. 2. 3 : 1, 2.
Annals of Mathematics. Ser. 2. 35 : 1, 2, 3.
Berichte über die Verhandlungen der Akad. der Wiss zu Leipzig: Math.-Phys. Klasse. 85 : 3, 4, 5; 86 : 1.
Boletin Matematico. 6 : 8.
Boletin Matematico Elemental.
Boletín del Seminario Matemático Argentino. 3 : 13, 14.
Bollettino della Unione Matematica Italiana. 13 : 1, 2, 3.
Bulletin de l'Académie Royale Serbe. A. 1.
Bulletin of the American Mathematical Society. 39 : 12; 40 : 1, 2, 3, 4, 5, 6, 7, 8.
Bulletin of the Calcutta Mathematical Society. 25 : 2, 3.
Communications . . . de Kharkow et . . . de l'Ukraine. Ser. 4. 7.
Contribución al Estudio de las Ciencias Fisicas y Matemáticas.
L'Enseignement Mathématique. 32 : 1-2, 3-4, 5-6.
Esercitazioni Matematiche (Catania). Ser. 2. 7 : 1-2, 3, 4, 5, 6-8, 9-10.
Gazeta Matematica. 39 : 5, 6, 7, 8, 9, 10, 11, 12.
Jahresbericht der Deutschen Mathematiker-Vereinigung. 43 : 5-8, 9-12.
Japanese Journal of Mathematics. 10 : 1, 2, 3, 4.
Journal of the Faculty of Science, Hokkaido. Ser. 1. 2 : 1-2.
Journal of the Indian Mathematical Society. N.S. 1 : 1.
Journal of the London Mathematical Society. 9 : 1, 2, 3.
Journal of the Mathematical Association of Japan. 15 : 6; 16 : 1, 2, 3, 4.
Mathematical Notes.
Mathematics Student. 1 : 4; 2 : 1.
Mathematics Teacher. 27 : 1, 2, 3, 4, 5.
Memoria (Univ. Nac. de la Plata).
Monatshefte für Mathematik und Physik.
Nieuw Archief voor Wiskunde. 18 : 2.
Periodico di Matematiche.
Proceedings of the Edinburgh Mathematical Society. Ser. 2. 4 : 1.
Proceedings of the Physico-Mathematical Society of Japan. Ser. 3. 16 : 1, 2, 3, 4, 5, 6, 7, 8.
Publicaciones . . . Fisico-Matemáticas . . . de la Plata.
Publications Mathématiques de Belgrade. 2.
Publications de la Faculté des Sciences de Masaryk. 188, 190.
Revista de Ciencias (Peru). 409-413.
Revista Matemática Hispano-Americana (Madrid). Ser. 2. 8 : 7, 8-9.
Revue Semestrielle des Publications Mathématiques. 38 : 6; 39 : 1, 2.
School Science and Mathematics. 34 : 1, 2, 3, 4, 5, 6.

Scripta Mathematica. **2** : 2, 3.
Sitzungsberichte der Berliner Mathematischen Gesellschaft. **33** : 1.
Studia Mathematica. 4.
Universidad de Zaragoza. Sección de Ciencias. **2** : 3, 4.
Unterrichtsblätter für Mathematik und Naturwissenschaften. **40** : 1, 2, 3, 4, 5, 6.
Wiskundige Opgaven met de Oplossingen. **16** : 2.

BOOKS RECEIVED FOR REVIEW.

W. G. Borchardt. *A compact arithmetic.* With answers. Pp. vii, 295, xxxviii. 3s. 1934. (Rivingtons)

W. G. Borchardt and A. D. Perrott. *A shorter trigonometry.* Pp. viii, 238, xxxii, xxxi. 4s.; without tables, 3s. 6d. 1934. (Bell)

M. C. Champneys. *An English bibliography of examinations. (1900-1934.)* Pp. xxiv, 141. 5s. 1934. International Institute Examinations Inquiry. (Macmillan)

F. Enriques e O. Chisini. *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche. IV. Funzioni ellittiche e abeliane.* Pp. viii, 274. L. 60. 1934. (Zanichelli, Bologna)

R. Estève et H. Mitault. *Éléments de Géométrie plane. I. La droite et le cercle.* Pp. vi, 144. 14 fr. 1934. (Gauthier-Villars)

H. Galbrun. *Théorie mathématique de l'Assurance Maladie.* Pp. viii, 218. 60 fr. 1934. E. Borel's *Traité du Calcul des Probabilités et de ses Applications,* Tome III, Fascicule 6. (Gauthier-Villars)

R. C. Gray. *Elementary dynamics for students of science and engineering.* Pp. xi, 211. 5s. 1934. (Macmillan)

E. Hopf. *Mathematical Problems of Radiative Equilibrium.* Pp. viii, 105. 6s. 1934. Cambridge Tracts, 31. (Cambridge)

C. E. Inglis. *A mathematical treatise on vibrations in railway bridges.* Pp. xxv, 203. 21s. 1934. (Cambridge)

H. C. Ives. *Mathematical tables.* Pp. vii, 160. 8s. 6d. 1934. (John Wiley and Sons; Chapman and Hall)

Sir James Jeans. *The new background of science.* 2nd edition. Pp. viii, 312. 7s. 6d. 1934. (Cambridge)

C. Juel. *Vorlesungen über projektive Geometrie mit besonderer Berücksichtigung der v. Staudschen Imaginärtheorie.* Pp. xi, 287. RM. 21; geb. RM. 22.50. 1934. Grundlehren der math. Wiss., 42. (Springer)

W. F. Kern and J. R. Bland. *Solid mensuration.* Pp. viii, 73. 7s. 6d. Wrappers. 1934. (John Wiley and Sons; Chapman and Hall)

E. Lindelöf. *Einführung in die höhere Analysis.* Translated from the first Swedish and second Finnish editions by E. Ullrich. Pp. ix, 526. Geb. RM. 16. 1934. (Teubner)

H. M. Macdonald. *Electromagnetism.* Pp. xv, 178. 12s. 6d. 1934. (Bell)

F. S. Nowlan. *Analytic geometry.* 2nd edition. Pp. xii, 352. 13s. 6d. 1934. (McGraw-Hill)

H. B. Phillips. *Differential equations.* 3rd edition. Pp. vi, 125. 10s. 6d. 1934. (John Wiley and Sons; Chapman and Hall)

V. C. Poor. *Analytical geometry.* Pp. v, 244. 13s. 6d. 1934. (John Wiley and Sons; Chapman and Hall)

G. Scheffers. *Wie findet und zeichnet man Gradnetze von Land- und Sternkarten?* Pp. 98. Kart. RM. 2.40. 1934. Mathematisch-Physikalische Bibliothek, Reihe I, Bd. 85/86. (Teubner)

W. F. F. Shearcroft and G. W. Spriggs. *Post-primary mathematics. I.* Pp. 172. *II.* Pp. 207. *III.* Pp. 213. 2s. 6d. each; without answers, 2s. each. 1934. (Harrap)

T. Y. Thomas. *The differential invariants of generalized spaces.* Pp. x, 241. 21s. 1934. (Cambridge)

W. M. Venable. *The sub-atoms.* Pp. viii, 248. 9s. 1933. (Williams and Wilkins, Baltimore; Baillière, Tindall and Cox)

Rapid arithmetic calculations. III. Household and business. Pp. vii, 47, 16. Without answers, 4d. and 6d.; with answers, 6d. and 8d. 1934. (Oxford)

LONDON BRANCH.

President: A. R. Hinks, Esq. (Gresham Professor of Astronomy).

Chairman: F. C. Boon (Dulwich College).

Programme for 1934-1935.

All meetings are held at Bedford College, Regent's Park, N.W. 1, on Saturdays, at 3 p.m. They take the form of a paper followed by an informal discussion, and tea (1s. a head). Members of the Association, and other visitors, are welcome at all meetings.

1934.

Oct. 27th. "Teaching Plan and Elevation to Beginners in Geometry."—A. W. Riley (Boys' Central School, Stroud).

Nov. 24th. Presidential Address: "The Figure of the Earth."—(This lecture will be illustrated by lantern slides.)

1935.

Jan. 26th. Meeting for Annual Business and Discussion of Members' Topics. Suggestions for topics are invited.

Feb. 23rd. "The teaching of Mathematical Analysis in Schools."—N. R. C. Dockeray (Harrow School).

Mar. 23rd. "Experimenting with a Junior Mathematical Association in Schools."—G. L. Parsons (Merchant Taylors' School).

A visit to the National Physical Laboratory will be arranged for the summer of 1935.

Secretaries: Miss E. L. Barnard (Godolphin and Latymer School), 26 Blomfield Road, W. 9.
C. T. Daltry (Roan School, Greenwich), 112 Canberra Road, S.E. 7.

GAZETA MATEMATICA

On resuming exchange after an interval of ten years, the Librarian was able not only to make the exchange retrospective to the point at which contact was lost but also to fill an earlier gap, and the set of the Rumanian namesake of the *Gazette* is now perfect from the beginning in 1895, eighteen months after the first appearance of our own journal.

REVUE SEMESTRIELLE

As reported in the *Gazette* at the time (vol. 16, p. x; 1932 July inset), the form of the *Revue Semestrielle des publications mathématiques* was changed completely on amalgamation with the *Jahrbuch über die Fortschritte der Mathematik.* The set of volumes in the old style has now been rounded off by a *Table des Matières* for vols. 31-36, similar to the six five-yearly index-volumes which facilitate search in the vols. 1-30.

The annual volume of the *Revue* is now composed of six numbers which appear at intervals of two months. Dare we ask if the Académie sanctions the retention of the original title.

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession; from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament thereunto."—BACON (Preface, *Maxims of Law*).

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports gifts as follows:

From Mr. **T. A. A. Broadbent,** a schoolbook by C. V. Durell and A. Robson, with:

H. Dörrie	Triumph der Mathematik One hundred famous problems.	1933
J. H. Jeans	New Background of Science {2}	1934
T. E. Jessop	Bibliography of George Berkeley With an inventory of manuscript remains, by A. A. Luce.	1934
W. Lietzmann	Lustiges und Merkwürdiges von Zahlen und Formen {4}	1930
W. F. Osgood	Advanced Calculus	1925
G. Prasad	Some Great Mathematicians of the Nineteenth Century; I, II	1933, 1934

From Mr. **F. J. Cock,** textbooks by E. G. Beck and W. P. Workman, with:

J. L. Coolidge	Probability	1925
T. Dantzig	Number, the Language of Science	1930
H. Freeman	Actuarial Mathematics	1931
J. H. Jeans	Astronomy and Cosmogony	1928
P. A. MacMahon	Combinatory Analysis (2 vols.)	1915, 1916
Tables for Statisticians and Biometricians. I {2} Edited by K. Pearson.		1924

From Mr. **T. M. A. Cooper,** schoolbooks by W. F. F. Shearcroft and G. W. Spriggs (3).

From Lt.-Col. **W. A. Garstin:**

M. E. C. Jordan	Cours d'Analyse {3} (3 vols.)	1909, 1913, 1915

From Mr. **S. Johnston,** a book on the Mannheim slide rule, and a textbook by P. Frost, with:

A. L. Baker	Quaternions	1911
G. A. Bliss	Calculus of Variations {1 (1925) rep.} Carus 1	1927
W. E. Byerly	Introduction to the Calculus of Variations {1 (1917) rep.}	1920
G. U. Yule	Introduction to the Theory of Statistics {6}	1922

From Prof. **E. H. Neville,** a set of Tables by F. W. Johnson.

From Mr. **A. B. Oldfield:**

E. Hopf	Mathematical Problems of Radiative Equilibrium Cambridge 31	1934

From Miss **B. T. Robins,** textbooks by S. L. Loney (2), J. J. Milne, C. Smith (2), I. Todhunter.

From the author:

B. A. Sueltz	The Status of Teachers of Secondary Mathematics in the United States A study made for the American Committee of the C.I.E.M.	1934

From Mr. **S. J. Tupper**:

H. Hill — Euclid's *Elements* - - - - - - - 1726

"The Six First, together with the Eleventh and Twelfth Books, . . ., demonstrated after a new, plain, and easie method."

From Prof. **G. N. Watson**:

J. Lense — Reihenentwicklungen in der mathematischen Physik 1933

From the Staff of Leeds Girls' High School, through Miss **M. E. Bowman** and Miss **N. G. Shapley**:

H. S. Carslaw — Non-Euclidean Geometry and Trigonometry - - 1916

S. P. Thompson — Calculus made Easy (2 (1914) rev.) - - - - 1920

This copy of a book whose startling success as an anonymous production in 1910 caused its authorship soon to be revealed, was imperfectly bound, and Messrs. Macmillan kindly sent a perfect copy.

together with schoolbooks by J. Alison and J. B. Clark, S. Barnard and J. M. Child (2), W. S. Beard, W. G. Borchardt, W. G. Borchardt and A. D. Perrott (2), S. E. Brown, V. S. Bryant, A. G. Cracknell, L. Crosland, C. Davison, C. Davison and C. H. Richards, R. Deakin, T. Dennis, F. W. Dobbs and H. K. Marsden, A. E. Donkin, R. C. Fawdry, S. N. Forrest, R. W. M. Gibbs (3), G. A. Gibson and P. Pinkerton, R. T. Glazebrook (3), J. H. Grace and F. Rosenberg, H. S. Hall and S. R. Knight, H. S. Hall and F. H. Stevens (2), W. Hunter, C. S. Jackson and W. M. Roberts, C. M. Jessop, C. M. Jessop and G. W. Caunt, A. C. Jones and P. H. Wykes, R. Lachlan and W. C. Fletcher, A. E. Layng, S. L. Loney (3), S. L. Loney and L. W. Grenville, A. Macgregor and J. D. Fulton, M. Maclean, H. J. Mann and J. S. Norman, F. M. Marzials, J. Milne, J. Milne and J. W. Robertson (2), B. C. Molony (2), A. Morley, J. S. Norman and F. K. Norman, A. H. E. Norris, T. P. Nunn, E. P. Oakes, R. S. Osborne, J. C. Pincock, H. C. Playne and R. C. Fawdry, F. F. Potter and V. L. Hilliard, F. F. Potter and F. C. Rice, A. J. Pressland and C. Tweedie, E. A. Price, F. A. J. Rivett, J. W. Robertson, G. Simmonds, E. L. Thorndike (2), H. M. Timpany, T. S. Usherwood and C. J. A. Trimble, C. J. L. Wagstaff, W. Watson, F. W. Westaway, W. P. Workman.

From the Facultad de Ciencias, Buenos Aires:

J. C. Vignaux — *Algunos puntos de la teoría de las series divergentes sumables* - - - - - - - - - 1933

From the London Mathematical Society:

Oxford, Cambridge, and Dublin Messenger of Mathematics, vol. 5 - 1871

An odd volume which completes the set, the earlier volumes having been given by Prof. Genese ten years ago. In 1871 the form of the *Messenger* was changed, and the wrappers bore the words "New Series" until publication ceased in 1929.

From the Mathematical Association of America:

J. W. Young — Projective Geometry - - - - - Carus 4 1930

From Miss **E. Cook**, Miss **E. M. Debenham**, and Miss **M. H. Greaves**, collections of back numbers of the *Gazette*.

The following have been bought:

R. F. A. Clebsch — Géométrie. III - - - - - - - - 1883

Edited by F. Lindemann. Translated from German into French by A. Benoist.

The other two volumes were already in the Library.

M. KRAITCHIK — Théorie des Nombres. II. - - - - - - 1926
An earlier volume, 1922, is now known as Tome I although it was not so called on the title page.

M. B. LEHMANN — Geometrische Aufbau gleichsummiger Zahlenfiguren 1932
"Neue mathematische Spiele für die Jugend."

W. J. WRIGHT — Determinants - - Modern Higher Mathematics 1 1875

Trilinear Coordinates - Modern Higher Mathematics 2 1877
Completing the set of Tracts issued by the author.

The Doctrine of Permutations and Combinations - - - - - 1795
A collection of reprints and essays, published by F. Maseres. *This copy replaces an imperfect copy described in List 2.*

A Table of Circles arising from . . . Division - - - - - - 1823
All the cycles that can occur in a recurring decimal equivalent to a fraction whose denominator is not greater than 1024, compiled by Henry Goodwyn. Regarded as a supplement to the author's *Tabular Series of Decimal Quotients*, 1823, which was given to the Library in 1926, this table enables the decimals in the *Series* to be extended to any number of places. The two tables were the subject of an article by J. W. L. Glaisher *On circulating decimals*, 1878.

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY.

As reported in the *Gazette* (Vol. XVII, p. xv; 1933 October inset), a change in this *Journal* was designed to take place after the end of 1932, the more elementary parts being budded off as the new *Mathematics Student*, now in its second volume. The *Journal* itself is in future to be a quarterly, and the first number reached us some time ago: New Series, vol. 1, no. 1; 1934. But 1933 was not a blank year for the *Journal*. On the contrary, vols. 1-14 were annual, 1909-1922, and each of vols. 15-19 covered two years; thus 1933 was the twenty-fifth year, and a special volume, prepared as a whole and not issued in parts, has appeared as vol. 20 to complete a series.

This "silver jubilee commemoration volume" consists of a record of the proceedings at the jubilee meeting held at Bombay, 21st-24th December, 1932, followed by twenty-two papers contributed by invitation. Of the papers, six only came from outside India; the Society did not issue invitations indiscriminately, but asked a few mathematicians who had previously shown interest in their doings; members of the Mathematical Association will be pleased to learn that the English contributors are their two most recent Presidents, Professor G. N. Watson, who deals with a number of formulae of the type of the famous Rogers-Ramanujan identities, and Professor E. H. Neville, who writes on iterative interpolation. Readers who have not been in touch with India will be amazed at the quality and at the range of the sixteen Indian papers. That charming and gentle scholar, Professor M. T. Naraniengar, editor from the foundation of the Club (as the Society was called for a few years) till 1927, to whom a well-deserved tribute was paid during the jubilee celebrations, recalls that at first there was hardly a contributor for whom he did not have to rewrite the manuscript and retrace the diagrams; when he compares the papers in this volume with the material out of which he fashioned the early numbers, he must enjoy, as only a modest man can properly enjoy, the knowledge that he has played a significant part in a great development; we add our congratulations very sincerely to those of his colleagues. Dr. Vaidyanathaswamy and Mr. Narasinga Rao have produced an interesting volume; the *Journal* and the *Student* are in good hands.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—Your reviewer has given a friendly welcome to my books on *Dynamics* published in 1929 and *Statics* published a few months ago, for which I am duly grateful.

I should not now venture to address you on the subject were it not that on p. 142 of the current volume of the *Gazette* it is suggested that I may hold a certain opinion which perhaps in honesty I ought to repudiate. "Mr. Ramsey may regret the distinction "—between " academic " and " practical "—" but his course is definitely ' academic '." Now, Sir, despite the fact that on opening the *Statics* at random I find four successive worked examples entitled " (i) *A cart wheel.* (ii) *A window sash with a broken cord.* (iii) *Braking of a carriage on an inclined plane.* (iv) *A pair of compasses*," yet I willingly accept your reviewer's judgment that the book is " academic ", and I am ready to agree that it contains little that is " practical ". But may I say that so far as I understand the distinction I greatly prefer the " academic " and regard it as much the more useful in teaching the principles of a subject ?

May I add a word on another small point upon which your reviewer comments : viz. the absence of any reference to the *cylindroid* ? It may be that some of your readers, who took the Tripos in the old days, do not realize the extent to which fashions changed with the abolition of the order of merit, and how, with the growth of new knowledge, many things which are of interest in themselves but not of great importance have had to give place. Examinations and courses of study at Cambridge now go hand in hand, and the cylindroid, though intrinsically beautiful, disappeared from Cambridge lectures and examinations about twenty-five years ago. It would have been rather misleading and in fact too " academic " at this date to have introduced it into a book mainly intended for schoolboys.—Your obedient servant,

Magdalene College, Cambridge. A. S. RAMSEY.

BOOKS RECEIVED FOR REVIEW.

G. H. Hardy, J. E. Littlewood and **G. Pólya.** *Inequalities.* Pp. xii, 314. 16s. 1934. (Cambridge)

R. C. Tolman. *Relativity Thermodynamics and Cosmology.* Pp. xv, 502. 30s. 1934. International series of monographs on physics. (Oxford)

J. de la Vaissière. *Méthodologie scientifique. Méthodologie Dynamique interne.* Pp. 109. 24 fr. 1933. (Beauchesne, Paris)

G. Verriest. *Évariste Galois et la théorie des équations algébriques.* Pp. 58. N.p. 1934. Reprinted from *Revue des Questions scientifiques.* (Gauthier-Villars)

H. Weyl. *Mind and Nature.* Pp. vii, 100. 6s. 6d. 1934. (University of Pennsylvania Press ; Oxford University Press)

F. Winter. *Das Spiel der 30 bunten Würfel. MacMahons Problem.* Pp. 128. Kart. RM. 3.60. 1934. (Teubner)

Abstracts of Dissertations for the degree of Doctor of Philosophy. Prepared by the Committee for Advanced Studies, University of Oxford. Pp. v, 303. 3s. 6d. 1934. (Oxford)

Correspondance du P. Marin Mersenne, Religieux Minime. I. 1617-1627. Edited by Mme. Paul Tannery. Pp. lxiv, 668. 120 fr. 1933. (Beauchesne, Paris)

Sir Isaac Newton's Mathematical Principles of Natural Philosophy and his System of the World. Translated into English by Andrew Motte in 1729. The translation revised, and supplied with an historical and explanatory appendix by Florian Cajori. Pp. xxxv, 680. 35s. 1934. (University of California Press ; Cambridge University Press)

INDEX

TO THE

MATHEMATICAL GAZETTE

VOL. XVIII.

FEBRUARY 1934, TO DECEMBER 1934

(Nos. 227–231)

TWO SHILLINGS AND SIXPENCE NET

LONDON

G. BELL AND SONS, LTD.

1934

E.
S.
R.
M.
G.
J.
H.
L.
T.

INDEX

TO THE

MATHEMATICAL GAZETTE

No. 227, FEBRUARY 1934—No. 231, DECEMBER 1934.

COMPILED BY MRS. T. A. A. BROADBENT.

1. Articles.
2. Mathematical Notes.
3. Reviews and Notices.
4. News of Branches.
5. Gleanings Far and Near.
6. The Pillory.
7. The Library.
8. Obituary Notices.
9. Correspondence.
10. Miscellaneous.

ARTICLES, ETC.

MATHEMATICAL NOTES.

REVIEWS AND NOTICES.

MISCELLANEOUS.

BETTER LATE THAN NEVER.

BRANCHES.

GLEANINGS FAR AND NEAR.

THE PILLORY.

THE LIBRARY.

DONATIONS.

NOTES.

OBITUARY.

CORRESPONDENCE.

MISCELLANEOUS.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

THE MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

LONDON

G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2

Vol. XVIII., No. 231. DECEMBER, 1934. 3s. Net.

CONTENTS.

The Mathematical Association.

THE ANNUAL MEETING will be held at the Institute of Education, Southampton Row, London, W.C. 1, on *Monday, 7th January*, 1935, at 2.15 p.m., and *Tuesday, 8th January*, 1935, at 10 a.m. and 2 p.m.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette".

Change of Address should be notified to C. Pendlebury, M.A., 39, Burlington Road, Chiswick, W.4. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

Subscriptions should be paid to Mr. W. H. Jex, 27 Marlborough Road, Chiswick, London, W.4.